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CONVERGENCE ACCELERATION OF CONTINUED FRACTIONS OF POINCARÉ'S TYPE

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This paper investigates the convergence properties of continued fractions of Poincaré's type with the aid of Pincherle's theorem, extends the investigation to the case where a tail is added to the approximants, and distinguishes between accelerative and decelerative tails. It shows that classical sequence transforms like θ or Levin/u algorithms accelerate the sequence of the approximants modified by the optimal accelerative tail. A two-step procedure is deduced which performs the acceleration of the convergence of continued fractions of Poincaré's type.

1. Introduction and notations

Let us consider the following continued fraction (cf):

$$\text{cf} = 1 + \frac{b_0}{a_1 +} \frac{b_1}{a_2 +} \cdots .$$

The associated recurrence is written as ($a_0 = 1$)

$$-b_k C_{k+1} + a_k C_k + C_{k-1} = 0.$$

It is well known that cf may be brought into the equivalent form

$$\text{cf} = 1 + \frac{\alpha_0}{1 +} \frac{\alpha_1}{1 +} \cdots , \tag{1}$$

and the associated recurrence becomes

$$-\alpha_k Z_{k+1} + Z_k + Z_{k-1} = 0. \tag{2}$$

This result is obtained by means of the following substitutions:

$$C_k = Z_k / \prod_{j=1}^{k} a_j \quad \text{and} \quad \alpha_k = b_k/(a_k a_{k+1}).$$

We shall call cf_k the kth approximant of cf: $\text{cf}_1 = 1$,

$$\text{cf}_k = 1 + \frac{\alpha_0}{1 +} \frac{\alpha_1}{1 +} \cdots \frac{\alpha_{k-1}}{1 +} \alpha_{k-2}.$$

If the sequence cf_k converges, then its limit is $\text{cf}_\infty = \text{cf}$. The speed of the convergence is accurately described by the parameter p defined via the relative error ($\text{cf} \neq 0$)

$$\exp(-p) = |\text{cf}_k - \text{cf}\,| / |\text{cf}\,|.$$

If cf equals zero, we may consider the absolute error $\exp(-\delta) = |\text{cf}_k - \text{cf}\,| = |\text{cf}_k|$.

2. Convergence property of continued fractions

Various monographs [1–4] dealing with cf present several convergence theorems valid for special cases. However, the main theorem on that topic is an early one due to Pincherle [5]. Its importance has been emphasized by Gautschi [6] in the context of the stable numerical calculation of the various approximants cf_k. We first proceed to show how the same theorem allows one to estimate the speed of convergence of cf_k.

2.1. Pincherle's theorem

(i) The cf (1) is convergent if and only if recurrence (2) possesses two contrasted linearly independent solutions X_k and Y_k (with X_k dominating Y_k as k increases and $Y_0 \neq 0$).

(ii) The exact value of cf is given by

$$\text{cf} = -Y_{-1}/Y_0.$$

(iii) The speed of the convergence is given by ($Y_{-1} \neq 0$, i.e. cf $\neq 0$)

$$\exp(-p) = -|Y_k/X_k|/|Y_{-1}| \quad \text{(with the choice } X_0 = 0 \text{ and } X_{-1} = 1).$$

If cf = 0, the absolute error is given by

$$\exp(-\delta) = |Y_k/X_k|/|Y_0| \quad \text{(with the choice } X_0 = 0 \text{ and } X_{-1} = 1).$$

In both cases the error is of the order of magnitude of the contrast factor, ρ, defined as

$$\rho_k = Y_k/X_k.$$

A good estimate of the error is obtained by replacing the exact solutions X_k and Y_k by their asymptotes. The problem of the determination of the asymptotes of the solutions of a linear recurrence has been studied theoretically by Birkhoff [7], Birkhoff and Trjitzinsky [8], Nörlund [9], Culmer [10], and Turrittin [11]. They found that in the case of recurrences of Poincaré's type, i.e. recurrences like (2), with α_k written as

$$\alpha_k \sim k^{\nu/r}\left[a_0 + a_1 k^{-1/r} + a_2 k^{-2/r} + \cdots\right], \qquad r \in \mathbb{Z}^+, \quad \nu \in \mathbb{Z},$$

which is only valid for large k, one has asymptotically,

$$Z_k \sim k!^{\lambda} a^k k^w (\ln k)^s \exp\left(\lambda_1 k^{1/m} + \lambda_2 k^{2/m} + \cdots + \lambda_{m-1} k^{(m-1)/m}\right)$$
$$\times \left[1 + \mu_1 k^{-1/m} + \mu_2 k^{-2/m} + \cdots\right],$$

with m(integer) $\leqslant r\mu$ with μ equal to the multiplicity of the considered root z of the characteristic equation.

Denef and Piessens [12] and Branders [13] have indicated how to calculate the parameters λ, a, w, s, λ_i, and μ_i. We [14] have published extended tables of those coefficients in the easiest cases and will systematically refer to them in the following.

2.2. Asymptotic behaviour of the solutions of recurrences of Poincaré's type 1

We shall investigate in detail the convergence properties of cf_k in the important case of an associated recurrence of Poincaré's type 1. In that case the coefficients of the cf are rational, thus we write for a large k,

$$\alpha_k = k^{\nu}\left[a_0 + a_1/k + a_2/k^2 + \cdots\right], \quad \nu \in \mathbb{Z},$$

and

$$-\alpha_k Z_{k+1} + Z_k + Z_{k-1} = 0.$$

The characteristic equation for k large is written as [13]

$$-a_0 k^\nu z^2 + z + 1 = 0.$$

Its roots are asymptotic to

$$z_{1,2} \sim \pm k^{-\nu/2}/\sqrt{a_0} \qquad \text{if } \nu > 0,$$
$$z_{1,2} \sim \left[1 \pm (1+4a_0)^{1/2}\right]/(2a_0) \qquad \text{if } \nu = 0,$$
$$z_1 \sim -1 \quad \text{and} \quad z_2 \sim k^{-\nu}/a_0 \qquad \text{if } \nu < 0.$$

Thus three distinct regimes must be studied separately by using the procedure described in [12,14]. The result of the complete discussion is presented in Table 1. Nine cases are distinguished, each exhibiting its own convergence behaviour. The results of the third column allow one to predict the type of convergence (or divergence), i.e. monotone or oscillatory, according to the sign of ρ_k and the rate of convergence.

Numerical examples

We have retained six interesting types of continued fractions among those presented in Table 1. The case for $\nu > 2$ has been excluded because the cf is divergent, and further we do not consider the case $\nu < 0$ because the cf is so rapidly convergent that no acceleration algorithm is superior to the simple forward calculation of the successive approximants:

$$\text{cf} = 1 + \frac{\alpha_0}{1} \, \frac{\alpha_1}{+ \, 1} \, + \cdots .$$

(1) cf1: $\alpha_k = -\frac{99}{400} - 0.001/(k+1)$, $\rho_k \sim \left(\frac{9}{11}\right)^k k^{-4/99}$.

(2) cf2: $\alpha_k = -\frac{1}{4} + 0.05/(k+1)$, $\rho_k \sim \exp(-1.79\sqrt{k})$.

(3) cf3: $\alpha_k = -\frac{1}{4} - 0.009/(k+1)^2$, $\rho_k \sim k^{-\sqrt{0.856}}$.

(4) cf4: $\alpha_k = -\frac{1}{16}(2k+5)^2/[(k+2)(k+3)]$, $\rho_k \sim 1/\ln k$

(5) cf5: $\alpha_k = 9(k+0.1)$, $\rho_k \sim (-1)^k \exp\left(-\frac{2}{3}\sqrt{k}\right)$.

(6) cf6: $\alpha_k = 1.15(k+0.5)^2$, $\rho_k \sim (-1)^k k^{-1/\sqrt{1.15}}$.

All these sequences converge more or less slowly and the record of slowness is held by cf4. After having performed some numerical experiments we report in Table 2 the following results: the limit of the cf (obtained by a technique which will be explained further on), the accuracy p of the 13th approximant cf_{13} (with $\text{cf}_1 = 1$, $\text{cf}_2 = 1 + \alpha_0, \ldots$), and the best accuracy p obtainable through three classical accelerative procedures: ε, θ, and Levin/u (where p is equal to the number of significant Napierian figures which are exact in the answer). These examples show that Brezinski's θ-algorithm and Levin's u-transform seem to be the most efficient with a slight

Table 1
Asymptotic expansion of the solutions of the associated recurrence and rate of convergence of the corresponding cf. $P(1/x)$, $P'(1/x)$, and $P''(1/x)$ denote polynomials in the variable $1/x$. The precision is given by: $p = -\ln|\rho_k|$

ν values	Dominant solution X_k and dominated solution Y_k	$\rho_k \sim Y_k/X_k$	Type of convergence
$\nu=0$ $1+4a_0 \neq 0$	$\left.\begin{matrix}X_k\\Y_k\end{matrix}\right\}\sim\left(\frac{1\pm\sigma}{2a_0}\right)^k k^{a_1(-1\pm 1/\sigma)/(2a_0)}\left\{\begin{matrix}P(1/k)\\P'(1/k)\end{matrix}\right.$ $\sigma=(1+4a_0)^{1/2}$	$\left(\frac{1-\sigma}{1+\sigma}\right)^k k^{-a_1/(a_0\sigma)}$	fast if $1+4a_0>0$ divergent if $1+4a_0<0$
$\nu=0$ $1+4a_0=0$ $a_1\neq 0$	$\left.\begin{matrix}X_k\\Y_k\end{matrix}\right\}\sim(-2)^k k^{2a_1+1/4}\exp[\pm 4(a_1k)^{1/2}]\left\{\begin{matrix}P(1/\sqrt{k})\\P'(1/\sqrt{k})\end{matrix}\right.$	$\exp[-8(a_1k)^{1/2}]$	good if $a_1>0$ divergent if $a_1<0$
$\nu=0$ $1+4a_0=0$ $a_1=0$ $1+16a_2\neq 0$	$\left.\begin{matrix}X_k\\Y_k\end{matrix}\right\}\sim(-2)^k k^{(1\pm\tau)/2}\left\{\begin{matrix}P(1/k)\\P'(1/k)\end{matrix}\right.$ $\tau=(1+16a_2)^{1/2}$	$k^{-\tau}$	slow if $1+16a_2=\tau>0$ divergent if $\tau<0$
$\nu=0$ $1+4a_0=0$ $a_1=0$ $1+16a_2=0$	$X_k\sim(-2)^k\sqrt{k}\ln kP(1/k)+(-2)^k\sqrt{k}P'(1/k)$ $Y_k\sim(-2)^k\sqrt{k}P''(1/k)$	$1/\ln k$	ultra slow
$\nu=-2,-3,\ldots$	$X_k\sim k^{\nu-a_1/a_0}k!^{-\nu}a_0^{-k}P(1/k)$ $\quad Y_k\sim(-1)^kP'(1/k)$	$(-a_0)^k k!^{\nu} k^{a_1/a_0-\nu}$	ultra fast if a_0 moderate
$\nu=-1$	$X_k\sim k^{a_0-a_1/a_0-1}k!a_0^{-k}P(1/k)$ $\quad Y_k\sim(-1)^kk^{-a_0}P'(1/k)$	$k^{1-2a_0+a_1/a_0}(-a_0)^k/k!$	fast if a_0 moderate
$\nu=1$	$\left.\begin{matrix}X_k\\Y_k\end{matrix}\right\}\sim k^{1/4-a_1/(2a_0)}k!^{-1/2}(\pm\sqrt{a_0})^{-k}\exp[\pm(k/a_0)^{1/2}]\left\{\begin{matrix}P(1/\sqrt{k})\\P'(1/\sqrt{k})\end{matrix}\right.$	$(-1)^k\exp[-2(k/a_0)^{1/2}]$	good if $a_0>0$ divergent if $a_0<0$
$\nu=2$	$\left.\begin{matrix}X_k\\Y_k\end{matrix}\right\}\sim k^{1/2-a_1/(2a_0)\pm 1/(2\sqrt{a_0})}(\pm\sqrt{a_0})^{-k}k!^{-1}\left\{\begin{matrix}P(1/k)\\P'(1/k)\end{matrix}\right.$	$(-1)^k k^{-1/\sqrt{a_0}}$	slow if $a_0>0$ divergent if $a_0<0$
$\nu=3,4,\ldots$	$\left.\begin{matrix}X_k\\Y_k\end{matrix}\right\}\sim k^{\nu/4-a_1/(2a_0)}k!^{-\nu/2}(\pm\sqrt{a_0})^{-k}\left\{\begin{matrix}P(1/k)\\P'(1/k)\end{matrix}\right.$	$(-1)^k$	oscillatory divergent

Table 2
Numerical tests performed on the approximants $cf_1 = 1, \ldots, cf_{13}$ of six test-cf: ε, θ, and Levin/u. The number of decimal figures is equal to $p/2.3$

cf	Limit	$p(cf_{13})$	$p(\varepsilon_{12}^{(0)})$	$p(\theta_8^{(0)})$	$p(u_{11}^{(0)})$
cf1	0.546179307131983	4.1	12.0	11.9	9.7
cf2	0.6822455410430423	6.0	10.2	14.4	13.1
cf3	0.469192286382208	2.3	3.5	15.8	10.9
cf4	0.375 (exact)	1.1	1.6	3.6	2.5
cf5	1.27992876522640	2.9	3.3	2.7	4.4
cf6	1.1508693968392	4.0	7.8	6.3	6.8

advantage for θ. However, the acceleration is not particularly impressive even in the most favourable case. As suspected the case of cf4 is especially slow.

2.3. *The reasons of the relative failure of the classical acceleration procedures*

We first recall the well-known algorithm which calculates the successive approximants of the cf (1): one calculates the sequences N_k and D_k obeying the associated recurrence (2) with the initial conditions:

$$N_{-1} = 0, \quad N_0 = 1 \quad \text{and} \quad D_{-1} = 1, \quad D_0 = 0. \tag{3}$$

One has

$$cf_k = N_k / D_k \quad \text{for } k = 1, 2, \ldots.$$

Pincherle's theorem allows the prediction of the asymptotic behaviour of the sequence cf_k. Indeed, because $Y_k = Y_{-1} D_k + Y_0 N_k$, one finds

$$cf_k - cf = (Y_k / D_k)/Y_0 = (Y_{-1}/Y_0) \frac{1}{1 - Y_0 N_k / Y_k}.$$

This shows that $cf_k - cf$ may in general be expressed in the nonlinear form

$$cf_k - cf \sim 1/(\lambda_1 g_1(k) + \lambda_2 g_2(k) + \cdots), \tag{4}$$

where $g_{i+1}(k)$ is negligible relative to $g_i(k)$ as k increases.

The consequence is that the E-algorithm of Brezinski [15] and Havie [16] is not immediately applicable to our case: moreover, expanding the right-hand side of (4) into the linear form $\mu_1 \phi_1(k) + \mu_2 \phi_2(k) + \cdots$ is generally impossible without increasing the number of functions ϕ which will make the E-algorithm poorly efficient.

The fact that asymptotic behaviours like (4) do not belong to the kernels of ε, θ, and u entails the relative inefficiency of these algorithms for the problem considered.

For example, we have, for the six families investigated, and referring to the notations of Table 1:

$$\mathrm{cf1}_k - \mathrm{cf1} \sim 1/\left[\lambda + \left(\frac{1+\sigma}{1-\sigma}\right)^k k^{a_1/(a_0\sigma)}\left(\lambda_0 + \lambda_1 k^{-1} + \cdots\right)\right],$$

$$\mathrm{cf2}_k - \mathrm{cf2} \sim 1/\left[\lambda + \exp\left[8(a_1 k)^{1/2}\right]\left(\lambda_0 + \lambda_1 k^{-1/2} + \lambda_2 k^{-2/2} + \cdots\right)\right],$$

$$\mathrm{cf3}_k - \mathrm{cf3} \sim 1/\left[\lambda + k^{\tau}\left(\lambda_0 + \lambda_1 k^{-1} + \cdots\right)\right],$$

$$\mathrm{cf4}_k - \mathrm{cf4} \sim 1/\left[\lambda_0 + \mu_0 \ln k + \lambda_1 k^{-1} + \mu_1 k^{-1} \ln k + \cdots\right],$$

$$\mathrm{cf5}_k - \mathrm{cf5} \sim 1/\left[\lambda + (-1)^k \exp\left[2(k/a_0)^{1/2}\right]\left(\lambda_0 + \lambda_1 k^{-1/2} + \lambda_2 k^{-2/2} + \cdots\right)\right],$$

$$\mathrm{cf6}_k - \mathrm{cf6} \sim 1/\left[\lambda + (-1)^k k^{1/\sqrt{a_0}}\left(\lambda_0 + \lambda_1 k^{-1} + \cdots\right)\right],$$

where λ and λ_i are constant numbers.

Remark. In very special cases, these asymptotic behaviours may be rewritten as linear forms. For example, if we suppose that τ is a positive integer, we may write

$$\mathrm{cf3}_k - \mathrm{cf3} \sim k^{-\tau}\left(\mu_0 + \mu_1 k^{-1} + \mu_2 k^{-2} + \cdots\right),$$

and expect the full efficiency of θ or Levin/u on the sequence $\mathrm{cf3}_k$.

An obvious numerical example is furnished by the cf:

$$1 + \frac{\alpha_0}{1} + \frac{\alpha_1}{1} + \cdots,$$

with

$$\alpha_k = -(k+2)^4/\left[(2k^2+6k+5)(2k^2+10k+13)\right].$$

That cf corresponds to the partial sums of the series $S_k = \Sigma_{j=0}^{k} 1/(j+1)^2$, according to the formula $S_k = 1 + 1/(5\mathrm{cf}_k - 1)$.

We have

$$\alpha_k \sim -\tfrac{1}{4} + \tfrac{1}{16}k^{-4} + \cdots, \quad k \text{ large},$$

which corresponds to the case: $1 + 4a_0 = 0$, $a_1 = 0$, $(1 + 16a_2)^{1/2} = \tau = 1$. We conclude that

$$\mathrm{cf}_k - \mathrm{cf} \sim k^{-1}\left(\mu_0 + \mu_1 k^{-1} + \cdots\right).$$

It can be verified that the θ-algorithm accelerates cf_k as efficiently as with the classical sequence S_k.

3. An acceleration procedure valid for cf of Poincaré's type

This section is devoted to the determination of an alternative method for the acceleration of cf of Poincaré's type.

3.1. Convergence of cf with a tail

So far we have considered standard approximants of the type

$$\mathrm{cf}_k = 1 + \frac{\alpha_0}{1} + \frac{\alpha_1}{1} + \cdots + \frac{\alpha_{k-3}}{1} + \alpha_{k-2}, \quad k = 2, 3, \ldots, \qquad \mathrm{cf}_1 = 1.$$

We now consider the modified sequences

$$\mathrm{cf}_k^* = 1 + \frac{\alpha_0}{1} + \frac{\alpha_1}{1} + \cdots + \frac{\alpha_{k-3}}{1} + \frac{\alpha_{k-2}}{\phi_{k-1}}, \quad k = 2, 3, \ldots, \qquad \mathrm{cf}_1^* = 1,$$

where ϕ_{k-1} is a sequence called the tail of the cf.

If the sequence cf_k converges, it is immediately apparent that cf_k^* will converge to the same limit provided one chooses

$$\phi_{k-1} = 1 + \frac{\alpha_{k-1}}{1} + \frac{\alpha_k}{1} + \cdots.$$

A straightforward application of Pincherle's theorem leads to the exact value of the tail which makes $\mathrm{cf}_k^* = \mathrm{cf}$ for all values k:

$$\phi_k = -Y_{k-1}/Y_k.$$

We shall now try to elucidate how cf_k^* behaves when the tail is arbitrarily chosen.

Theorem 3.1. *The approximants* cf_k^* *are given by the following expression*:

$$\mathrm{cf}_k^* = (N_{k-2} + \phi_{k-1}N_{k-1})/(D_{k-2} + \phi_{k-1}D_{k-1}), \quad k = 2, 3, \ldots, \qquad \mathrm{cf}_1^* = 1,$$

where the sequences N_k *and* D_k *are defined as before* (*see* (3)).

In particular, if $\phi_{k-1} = 1$, *we recover the well-known formula* $\mathrm{cf}_k = N_k/D_k$.

Proof. cf_n^* is obtained by repeated application of the associated recurrence (2) to N_k and D_k for $k = 0, \ldots, n-1$. However, when $k = n-2$, α_{n-2} must be replaced by α_{n-2}/ϕ_{n-1}. More precisely, we may write

$$N_{n-3}^* = N_{n-3}, \qquad N_{n-2}^* = N_{n-2}, \qquad N_{n-1}^* = (N_{n-2}^* + N_{n-3}^*)\phi_{n-1}/\alpha_{n-2} = N_{n-1}\phi_{n-1},$$

$$D_{n-3}^* = D_{n-3}, \qquad D_{n-2}^* = D_{n-2}, \qquad D_{n-1}^* = (D_{n-2}^* + D_{n-3}^*)\phi_{n-1}/\alpha_{n-2} = D_{n-1}\phi_{n-1}.$$

Hence

$$N_n^* = (N_{n-1}^* + N_{n-2}^*)/\alpha_{n+1} = (N_{n-2} + N_{n-1}\phi_{n-1})/\alpha_{n+1},$$

$$D_n^* = (D_{n-1}^* + D_{n-2}^*)/\alpha_{n+1} = (D_{n-2} + N_{n-1}\phi_{n-1})/\alpha_{n+1},$$

and the result $\mathrm{cf}_n^* = N_n^*/D_n^*$ proves the theorem. □

3.1.1.

Let us look for the conditions under which cf_k^* converges to the same limit cf as cf_k. For that purpose we consider the expression

$$\mathrm{cf}_k^* - \mathrm{cf} = (N_{k-2} + \phi_{k-1}N_{k-1})/(D_{k-2} + \phi_{k-1}D_{k-1}) + Y_{-1}/Y_0.$$

By taking into account the evident fact that $Y_k = Y_0 N_k + Y_{-1}D_k$, we find

$$\mathrm{cf}_k^* - \mathrm{cf} = (1/Y_0)(Y_{k-2} + \phi_{k-1}Y_{k-1})/(D_{k-2} + \phi_{k-1}D_{k-1}).$$

Considering the dominant solution $U_k = D_k - Y_k$ the equation is rewritten as

$$\mathrm{cf}_k^* - \mathrm{cf} = (1/Y_0)\left[1\Big/\left(1 + \frac{U_{k-2} + \phi_{k-1}U_{k-1}}{Y_{k-2} + \phi_{k-1}Y_{k-1}}\right)\right]$$
$$= (1/Y_0)[1/(1 + \xi_k)],$$

where we have set $\xi_k = (U_{k-1}/Y_{k-1})(U_{k-2}/U_{k-1} + \phi_{k-1})/(Y_{k-2}/Y_{k-1} + \phi_{k-1})$. The behaviour of ξ_k for k large depends on the sequence ϕ_k. A complete discussion is as follows.

Lemma 3.2. *If U_k dominates Y_k, then $|U_{k-1}/U_k| < |Y_{k-1}/Y_k|$ for k large enough.*

Proof. Ad absurdum. Suppose that

$$|U_j/U_{j-1}| < |Y_j/Y_{j-1}| \quad \text{for } j > N.$$

Then for k sufficiently large we should have

$$|U_k/U_{N-1}| = \prod_{j=N}^{k} |U_j/U_{j-1}| < \prod_{j=N}^{k} |Y_j/Y_{j-1}| = |Y_k/Y_{N-1}|,$$

and it would be impossible that U_k dominates Y_k.

Note 3.3. $|U_{k-1}/U_k|$ and $|Y_{k-1}/Y_k|$ are not necessarily contrasted as $k \to \infty$.

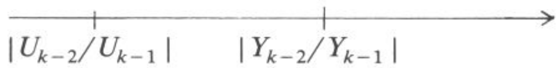
3.1.2.

The consequences of the lemma are as follows:

(1) If $|Y_{k-1}/Y_k|$ and $|U_{k-1}/U_k|$ are contrasted, i.e. if

$$\lim_{k\to\infty} |U_{k-1}Y_k/U_kY_{k-1}| = 0,$$

we distinguish among five cases:



(i) $|\phi_{k-1}|$ dominated by $|U_{k-2}/U_{k-1}|$: ξ_k behaves like U_{k-2}/Y_{k-2}; cf_k^* converges to cf like ρ_{k-2}, i.e. slightly less fast than cf_k.

(ii) $\phi_{k-1} \sim \kappa U_{k-1}/U_{k-1}$: If $\kappa \neq -1$, the conclusion of (i) remains valid. If $\kappa = -1$, we must add supplementary notation by writing $\phi_{k-1} = -U_{k-2}/U_{k-1}(1 + \eta(k))$. In that case ξ_k behaves like $-(U_{k-2}/Y_{k-2})\eta(k)$. If U_{k-2}/Y_{k-2} dominates $1/\eta(k)$, then cf_k^* converges to cf but the process is decelerated by a factor $\eta(k)$. If $1/\eta(k)$ dominates U_{k-2}/Y_{k-2}, then cf_k^* converges to a limit cf * distinct of cf, namely $\mathrm{cf}^* = \mathrm{cf} + 1/Y_0$. If $U_{k-2}/Y_{k-2} \sim \sigma/\eta(k)$, then cf_k^* converges to a limit $\mathrm{cf}^* \neq \mathrm{cf}$ if $\sigma \neq 1$, i.e. $\mathrm{cf}^* = \mathrm{cf} + (1/Y_0)/(1 + \sigma)$, and diverges otherwise.

(iii) ϕ_{k-1} intermediate between U_{k-2}/U_{k-1} and Y_{k-2}/Y_{k-1}: ξ_k behaves like $U_{k-1}\phi_{k-1}/Y_{k-2}$, i.e. $U_{k-2}/Y_{k-2} < \xi_k < U_{k-1}/Y_{k-1}$ so that cf_k^* converges to cf with a rate intermediate between ρ_{k-2} and ρ_{k-1}, i.e. hardly less fast than cf_k.

(iv) $\phi_k \sim \kappa Y_{k-2}/Y_{k-1}$: If $\kappa \neq -1$ one immediately sees that $\xi_k \sim (\kappa/(1 + \kappa))U_{k-1}/Y_{k-1}$, i.e. cf_k^* converges to cf like ρ_{k-1}. If $\kappa = -1$, we set $\phi_{k-1} = -(Y_{k-2}/Y_{k-1})(1 + \varepsilon(k))$, hence $\xi_k = (U_{k-1}/Y_{k-1})/\varepsilon(k)$. Because $\lim_{k\to\infty}\varepsilon(k) = 0$, we conclude that cf_k^* converges to cf like $\rho_{k-1}\varepsilon(k)$. i.e. the sequence is accelerated by a factor $\varepsilon(k)$.

(v) ϕ_{k-1} dominates Y_{k-2}/Y_{k-1}: $\xi_k \sim U_{k-1}/Y_{k-1}$ and cf_k^* converges to cf like ρ_{k-1}.
(2) A similar discussion holds if U_{k-2}/U_{k-1} and Y_{k-2}/Y_{k-1} are not contrasted.

3.1.3. Conclusions

We summarize the results in the following way. The two possibilities for a tail ϕ_k to modify the speed of convergence of a convergent cf is that ϕ_k be of the same order of magnitude as

$$u_k = U_{k-1}/U_k \quad \text{or} \quad y_k = -Y_{k-1}/Y_k.$$

- If u_k and y_k are contrasted, the convergence is
  - decelerated by a factor $\eta(k)$ if $\phi_k \sim u_k(1+\eta(k))$,
  - accelerated by a factor $\varepsilon(k)$ if $\phi_k \sim y_k(1+\varepsilon(k))$ (ε and $\eta \neq 0$).
- If u_k and y_k are not contrasted so that one can write

$$\phi_k \sim u_k(1+\eta(k)) \sim y_k(1+\varepsilon(k))$$

(with $\varepsilon(k)$ and $\eta(k) \neq 0$ and contrasted), then the convergence is accelerated by a factor $\varepsilon(k)/\eta(k)$ if η dominates ε and is decelerated by the same factor otherwise.

When the deceleration is too fast the convergence may be dramatically altered leading to a convergence to another limit or even to divergence.

In particular, if $\phi_k = -(\lambda U_{k-1} + \mu Y_{k-1})/(\lambda U_k + \mu Y_k)$, i.e. if ϕ_k is built as the exact ratio of two successive terms of an arbitrarily chosen dominant solution of the associated recurrence, one finds ($\lambda \neq \mu$)

$$\mathrm{cf}_k^* - \mathrm{cf} = (1/Y_0)\lambda/(\lambda-\mu).$$

If $\lambda = \mu$, the sequence cf_k^* does not exist. In other words, the choice $\phi_k = -D_{k-1}/D_k$ is forbidden (where D_k is defined by (3)).

3.2. Acceleration procedure for cf of Poincaré's type

We have seen that the tail ϕ_k produces an acceleration in the convergence of the approximants provided it is chosen in agreement with

$$\phi_k \sim -(Y_{k-1}/Y_k)(1+\varepsilon(k)),$$

where $\varepsilon(k)$ decreases to zero as k increases.

If by chance we know the exact value of the dominated solution Y_k, then the approximants will be exact: $\mathrm{cf}_k^* = \mathrm{cf}$. In all other cases it may be sufficient to take for ϕ_k the asymptotic behaviour of $-Y_{k-1}/Y_k$. That behaviour can be determined in a straightforward way in the case of a cf of Poincaré's type 1: we have presented in Table 1 the asymptotic expansions of the dominated solution Y_k which are valid in this case. For the sake of clarity let us detail the example of the ultra slow cf corresponding to the parameter values

$$\nu = 1 + 4a_0 = a_1 = 1 + 16a_2 = 0.$$

We see in Table 1 that $Y_k \sim (-2)^k\sqrt{k}\,P(1/k)$. The optimal tail behaves thus like

$$\phi_k \sim \tfrac{1}{2}(1-1/k)^{1/2}P(1/(k-1))/P(1/k)$$
$$\sim p_0 + p_1/k + p_2/k^2 + \cdots.$$

Table 3
Asymptotic behaviour of the optimal tail of cf of Poincaré's type 1

ν-values	Asymptotic behaviour of $\phi_k \sim -Y_{k-1}/Y_k$	Coefficients	ν-values	Asymptotic behaviour of $\phi_k \sim -Y_{k-1}/Y_k$	Coefficients
$\nu=0$ $1+4a_0 \neq 0$	$\sum_{i=0}^{\infty} p_i/k^i$	$p_0=\frac{1}{2}(1+(1+4a_0)^{1/2})=Q$ $p_1=a_1/(2Q-1)$ $p_2=(a_2-p_1^2-p_1+p_1Q)/(2Q-1)$ $p_3=(a_3-2p_1p_2+2p_2Q-2p_2+p_1^2$ $-p_1Q+p_1)/(2Q-1)$ $p_4=(a_4-2p_1p_3+3p_3Q-3p_3-p_2^2$ $+3p_1p_2-3p_2Q+3p_2-p_1^2$ $+p_1Q-p_1)/(2Q-1)$	$\nu=-2$	$\sum_{i=0}^{\infty} p_i/k^i$	$p_0=1$ $p_1=0$ $p_2=a_0$ $p_3=a_1$ $p_4=a_2-a_0^2$
$\nu=0$ $1+4a_0=0$ $a_1 \neq 0$	$\sum_{i=0}^{\infty} p_i/k^{i/2}$	$p_0=\frac{1}{2}$ $p_1=\sqrt{a_1}=Q$ $p_2=-\frac{1}{8}$ $p_3=(64a_2+32a_1+3)/(128Q)$ $p_4=-p_3/(4Q)$ $p_5=(64a_3-24a_1-48p_4-64p_3^2$ $+128p_3Q-3)/(128Q)$ $p_6=(-64p_5-128p_3p_4+160p_4Q$ $+40p_3+Q)/(128Q)$	$\nu=-1$	$\sum_{i=0}^{\infty} p_i/k^i$	$p_0=1$ $p_1=a_0$ $p_2=a_1-a_0^2$ $p_3=a_2-2p_1p_2+p_1^2$ $p_4=a_3-2p_1p_3-p_2^2+3p_1p_2-p_1^2$
$\nu=0$ $1+4a_0=0$ $a_1=0$ $1+16a_2 \neq 0$	$\sum_{i=0}^{\infty} p_i/k^i$	$p_0=\frac{1}{2}$ $p_1=\frac{1}{4}(-1+(1+16a_2)^{1/2})=Q$ $p_2=(a_2+a_3)/(2Q+1)$ $p_3=(2a_4-2a_2-2p_2^2+6p_2Q+3p_2)$ $/(4Q+3)$ $p_4=(a_5+a_2-2p_2p_3+4p_3Q+3p_3$ $+2p_2^2-4p_2Q-2p_2)/(2Q+2)$	$\nu=1$	$k^{1/2}\sum_{i=0}^{\infty} p_i/k^{i/2}$	$p_0=\sqrt{a_0}=Q$ $p_1=\frac{1}{2}$ $p_2=(4a_1-2a_0+1)/(8Q)$ $p_3=\frac{1}{8}$ $p_4=(8a_2+a_0-8p_2^2)/(16Q)$ $p_5=-p_2/(4Q)$ $p_6=(64a_3-4a_0-128p_2p_4+64p_4Q$ $+32p_2^2-16p_2Q-5)/(128Q)$
$\nu=0$ $1+4a_0=0$ $a_1=0$ $1+16a_2=0$	$\sum_{i=0}^{\infty} p_i/k^i$	$p_0=\frac{1}{2}$ $p_1=-\frac{1}{4}$ $p_2=\frac{1}{8}(16a_3-1)$ $p_3=\frac{1}{16}(16a_4-16p_2^2+12p_2+1)$ $p_4=\frac{1}{24}(16a_5-32p_2p_3+32p_3$ $+32p_2^2-16p_2-1)$	$\nu=2$	$k\sum_{i=0}^{\infty} p_i/k^i$	$p_0=\sqrt{a_0}=Q$ $p_1=(a_1-a_0+Q)/(2Q)$ $p_2=(a_2-p_1^2-p_1Q+p_1+Q)/(2Q)$ $p_3=(a_3-2p_1p_2+p_2)/(2Q)$ $p_4=(a_4-2p_1p_3+p_3Q+p_3-p_2^2$ $+p_1p_2-p_2Q-p_2)/(2Q)$

The coefficients p_i are determined by identification of the coefficients of the lowest powers of $1/k$ in the recurrence

$$1 - \phi_k + \alpha_k/\phi_{k+1} = 0,$$

with

$$\alpha_k = -\tfrac{1}{4} - \tfrac{1}{16}k^{-2} + a_3 k^{-3} + \cdots .$$

We find in detail:

$$1 - \left(p_0 + p_1/k + p_2/k^2 + \cdots \right)$$
$$+ \left(-\tfrac{1}{4} - \tfrac{1}{16}/k^2 + a_3/k^3 + \cdots \right) / \left(p_0 + p_1(k+1)^{-1} + \cdots \right) = 0.$$

In the first order we get $p_0^2 - p_0 + \frac{1}{4} = 0$, i.e. $p_0 = \frac{1}{2}$. In the second order: $p_1 = -\frac{1}{4}$. In the third order: $p_2 = -\frac{1}{8}(1 - 16a_3)$, and so on.

This kind of calculation is most easily performed by using a computer algebra system like REDUCE. We have reported in Table 3 the results of the complete calculations in the case of cf with rational coefficients (i.e. Poincaré's type 1).

Numerical Example 3.4. It is interesting to test the method on the super slow cf4 = 0.375, which corresponds to $\alpha_k = -\frac{1}{16}(2k+5)^2/((k+2)(k+3))$.

We have asymptotically $\alpha_k \sim -\frac{1}{4} - \frac{1}{16}k^{-2} + \frac{5}{16}k^{-3} - \frac{19}{16}k^{-4} + \cdots$. Hence, $a_0 = -\frac{1}{4}$, $a_1 = 0$, $a_2 = -\frac{1}{16}$, $a_3 = \frac{5}{16}$, $a_4 = -\frac{19}{16}, \ldots$. Looking at Table 3 we calculate the coefficients p_i which appear in

$$\phi_k = p_0 + p_1/k + p_2/k^2 + \cdots .$$

We find $p_0 = \frac{1}{2}$, $p_1 = -\frac{1}{4}$, $p_2 = \frac{1}{2}$, $p_3 = -1$. If we consider the sequence $\mathrm{cf}_1^*, \ldots, \mathrm{cf}_{13}^*$, built with the aid of the tail $\phi_k = p_0 + \cdots + p_3/k^3$, we find $\mathrm{cf}_{13}^* = 0.374418$ ($p = 6.4$). The number of exact figures has been multiplied by a factor of 6. But the important point is that the modified sequence is strongly accelerated by both θ and Levin/u algorithms. Applying θ to $\mathrm{cf}_1^*, \ldots, \mathrm{cf}_{13}^*$ leads to $\theta_8^{(0)}(\mathrm{cf4}^*) = 0.375000000613$ ($p = 20.2$), whereas applying Levin/u gives $u_{11}^{(0)}(\mathrm{cf4}^*) = 0.374999999405$ ($p = 20.2$).

It is an experimental fact that θ and Levin/u accelerate the convergence of the sequence of the approximants of a cf modified by the addition of an asymptotically optimal tail. We have tested this new procedure on the six test cases of cf of Table 2 with a tail which is exact up to the order k^{-3} (included). The results are displayed in Table 4, where it is seen that the method works efficiently and leads to an accuracy which is between four and twenty times the original accuracy for the chosen examples. The efficiencies of both θ and Levin/u are of equal order of magnitude, whereas ε is less efficient except for the cases cf5 and cf6.

3.3. Examples of application in physics

The same procedure may be used to accelerate the convergence of generalized continued fractions which are associated with n-term recurrence relations of Poincaré's type coefficients. A widely studied problem in physics is the calculation of the eigenvalues of Schrödinger's equation (SE) through the so-called Hill method. For example, the confinement potential $ax^2 + bx^4 + cx^6$

Table 4
The approximants $cf_1,\dots,cf_{13}$ have been modified by the introduction of an asymptotically correct tail ($p_0 + \cdots + p_3 k^{-3}$), leading to $cf_1^*,\dots,cf_{13}^*$. The new sequences have been accelerated by ε, θ, and Levin/u. The Napierian precision is given in each case for cf_{13}, cf_{13}^* and for the result of the transformation of the modified sequence cf *

cf *	$p(cf_{13})$	$p(cf_{13}^*)$	$p(\varepsilon)$	$p(\theta)$	$p(L/u)$
cf1*	4.1	10.5	14.8	20.8	19.3
cf2*	6.0	13.6	17.5	23.6	22.4
cf3*	2.3	18.3	21.2	30.0	27.6
cf4*	1.1	6.4	8.7	20.2	20.2
cf5*	2.9	15.7	26.4	27.1	28.3
cf6*	4.0	15.3	25.9	26.7	27.6

($c > 0$), has been extensively studied [17,18] in this context. The problem deals with the following SE:

$$\psi'' + (E - ax^2 - bx^4 - cx^6)\psi = 0,$$

with the limiting conditions $\psi(\pm\infty) = 0$.

Trying the solution ψ under the expanded form (even states)

$$\psi = \exp\left(-\tfrac{1}{4}\alpha x^4 + \tfrac{1}{2}\beta x^2\right) \sum_{k=0}^{\infty} C_k x^{2k},$$

where $\alpha = c^{1/2} > 0$ and $\beta = -\tfrac{1}{2}bc^{-1/2}$, we find that the C_k obey the second-order recurrence relation

$$(2k+1)(2k+2)C_{k+1} + [E + \beta(4k+1)]C_k + [\beta^2 - a - (4k-1)\alpha]C_{k-1} = 0,$$

with $k = 0, 1, 2,\dots$ and $C_{-1} = 0$.

If $b > 0$, the eigenvalues are deduced as the roots of the associated cf.

The general solutions C_k are easily found asymptotic to

$$C_k^{1,2} \sim (\pm c^{1/4})^k \Gamma(k)^{-1/2} k^w \exp\left(\pm\tfrac{1}{2}bc^{-3/4}k^{1/2}\right),$$

and the contrast between $C_k^{(1)}$ and $C_k^{(2)}$ may be written as

$$\rho_k \sim \exp(-p) \sim (-1)^k \exp\left(-bc^{-3/4}k^{1/2}\right).$$

The convergence of the cf appears to be slow, especially when $bc^{-3/4} < 1$. It may be accelerated by the procedure described in this paper.

4. Conclusion

This paper has mainly dealt with cf with rational coefficients. The convergence conditions have been emphasized and the rate of convergence has been given in each case. When the convergence appeared to be slow, we have shown that no classical acceleration procedure seemed to be particularly efficient. This was caused by the rather special asymptotic behaviour of the successive approximants.

We have discussed the effects of modifying the approximants by the introduction of an appropriate tail. This idea is not entirely new. Jacobsen [19] refers to it in a recent paper which contains a full bibliography on the topic. By reinvestigating it in the context of Pincherle's theorem, we have shown that the exact tail is $\phi_k = -Y_{k-1}/Y_k$, and that $\phi_k = -Y_{k-1}/Y_k(1+\varepsilon(k))$ is an accelerative tail provided that ε decreases sufficiently rapidly as k increases. Because the exact expression for Y_k is generally not known we have suggested the replacement of Y_{k-1}/Y_k by its asymptotic behaviour. Various methods could certainly be used for that purpose: we have presented an algebraic procedure which expands the tail in successive powers of $1/k$ or $1/\sqrt{k}$ depending on the case. It is particularly interesting to note that both θ and Levin/u transforms do accelerate the sequences of approximants modified in this way. The resulting two-step procedure appears to furnish an acceptable method for the acceleration of slowly convergent cf. This is clearly demonstrated by six numerical examples referred to as cf1,...,cf6 in the text. The example of the super slow cf4 is especially demonstrative. Our procedure strongly depends on the possibility of determining an asymptotic expansion for the optimal tail of the cf. This is quite straightforward if the coefficients of the cf are rational. It may also be shown that the procedure remains valid if the coefficients of the cf are expandable under the form

$$\alpha_k \sim k^{\nu/r}\left(p_0 + p_1 k^{-1/r} + p_2 k^{-2/r} + \cdots\right), \qquad \nu \in \mathbb{Z}, \quad r \in \mathbb{Z}^+,$$

corresponding to the so-called cf of Poincaré's type r.

The reader is referred to our earlier paper [14], where extended tables are published which describe the asymptotic behaviour of the dominated solution of the associated recurrence (2). The optimal tail results once again from the formula $\phi_k \simeq -Y_{k-1}/Y_k$.

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