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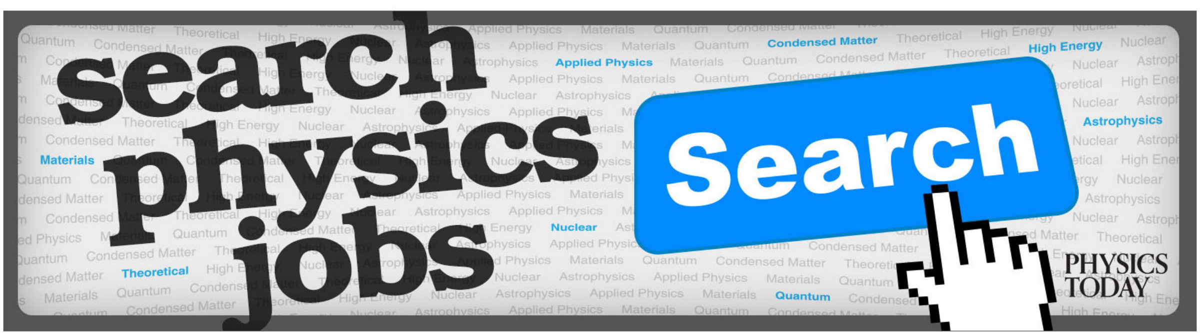

Motion in radial magnetic fields

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We study the problem of the motion of a charged particle in radial magnetic induction fields of the type $\mathbf{B} = h(\theta)\mathbf{r}/r^3$. A Coulomb electric field is also added. The mechanics of Newton, Einstein, Schrödinger, and Dirac are successively considered in view of exact solutions. The analogies between the four treatments are emphasized.

One way for studying mathematically the fundamental equations of mechanics is to put definite problems in view of *exact* solutions. When one considers the most complicated forces acting on a charged particle which allow an exact integration it is possible to realize exactly the limitations imposed by the actual advancement of analysis. Naturally it is another task to discover whether or not these forces (sometimes rather fantastic) are of interest for physicists. We have neglected this aspect here considering only in this first approach the performing of exact calculations which seem to stay at the limit of what is presently mathematically possible.

Of course such a problem can never be solved in all its generality. It is always necessary to treat separately well distinct classes of potentials. A few exact resolutions in classical mechanics are known[1,2]; in quantum mechanics the problem has been studied by various authors,[3–5] each bringing its own contribution after exposing the presumed complete bibliography. The present paper deals with the search of the general characteristics of the movement of a charged particle in magnetic induction fields of the type

$$\mathbf{B} = h(\theta)\mathbf{r}/r^3 \tag{1}$$

written in spherical coordinates r, θ, φ.

The problem is considered successively in the four fundamental mechanics: Newton's, Einstein's, Schrödinger's, and Dirac's. In view of increasing the generality (and the complexity!) of the calculations we consider that the particle simultaneously experiences an attractive Coulomb electric field. The analogies between the four treatments will be emphasized: it will be shown that they are of a rather strange character.

Notations: We shall denote ϵ and μ_0 the charge and the rest mass of the particle; the relativistic mass will be $\mu = \beta\mu_0$, where $\beta = (1 - v^2/c^2)^{-1/2}$. The Coulomb electric potential is $-(\mu_0/\epsilon)(H/r)$. $\mathbf{v}$ is the velocity of the particle and $\boldsymbol{\gamma}$ its acceleration, i is the symbol of complex numbers.

I. NEWTON'S MECHANICS

Newton's equation is written as

$$\boldsymbol{\gamma} = (\epsilon/\mu_0)\mathbf{v} \wedge h(\theta)\mathbf{r}/r^3 - H\mathbf{r}/r^3. \tag{2}$$

A. The radial integration

The conservation of energy implies

$$v^2 - 2H/r = a \quad (a = \text{const, negative for bound states}).$$

By scalar multiplication of (2) by $\mathbf{r}$ we get $\mathbf{r}\cdot\boldsymbol{\gamma} = -H/r$ from which we deduce $d(\mathbf{r}\cdot\mathbf{v})/dt = a + H/r$.

Remembering that $\mathbf{r}\cdot\mathbf{v} = 1/2\; dr^2/dt$ we have after a classical integration

$$\mathbf{r}\cdot\mathbf{v} = (ar^2 + 2\,Hr - b)^{1/2} \quad (b = \text{const}).$$

Finally we get

$$\int dt = \int r(ar^2 + 2Hr - b)^{-1/2}\, dr = I(r).$$

The radial motion is therefore described by the exact equation $t - t_0 = I(r) - I(r_0)$. In particular it is independent of the presence of the radial magnetic field. Therefore it coincides with the radial Kepler's movement.

B. The angular integrations

Let us define $\mathbf{P} = \mathbf{r} \wedge \mathbf{v}$ and let us calculate P^2. Denoting α the angle between $\mathbf{r}$ and $\mathbf{v}$ we have

$$P^2 = r^2v^2\sin^2\alpha$$

and

$$(\mathbf{r}\cdot\mathbf{v})^2 = r^2v^2\cos^2\alpha = ar^2 + 2Hr - b.$$

Adding these two results and taking into account the conservation of energy we arrive at $P^2 = b(>0)$. Thus the constant $\mu_0^2\, b$ is the total angular momentum squared. On another side one has in spherical coordinates:

$$P^2 = r^4(\dot\theta^2 + \sin^2\theta\dot\varphi^2) \tag{3}$$

The overdot denotes time differentiation.

Up to now the calculations are correct for every magnetic field of the species $\mathbf{B} = L(x, y, z)\mathbf{r}$, where the function L is arbitrary. Now we shall confine ourselves to the special form (1).

The field (1) derives from the following potential: $\mathbf{B} = \operatorname{curl} \mathbf{A}$ with

$$\mathbf{A} = [g(\theta)/r^2\sin\theta](-y, x, 0). \tag{4}$$

The functions h and g are connected by: $h(\theta) = g(\theta)\cot\theta + g'(\theta)$. Let us put

$$f(\theta) = (\epsilon/\mu_0)\int h(\theta)\sin\theta d\theta = (\epsilon/\mu_0)\, g(\theta)\sin\theta. \tag{5}$$

After vector multiplication of (2) by $\mathbf{r}$ one computes

$$dP_z/dt = (\epsilon/\mu_0)[h(\theta)/r][\mathbf{v}_z - (\mathbf{r}\cdot\mathbf{v})z/r^2].$$

Remembering that $z = r\cos\theta$ and $v_z = \dot r\cos\theta - r\sin\theta\dot\theta$ we deduce

$$P_z = d - (\epsilon/\mu_0)\int h(\theta)\sin\theta d\theta \quad (d = \text{const}).$$

But $P_z = r^2\sin^2\theta\dot\varphi$ so that we have

$$r^2\sin^2\theta\dot\varphi = d - f(\theta). \tag{6}$$

Equations (3) and (6) allow us to find the two last integrations needed for the complete solution:

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$$I=\int\frac{\sin\theta d\theta}{\{P^2\sin^2\theta-[d-f(\theta)]^2\}^{1/2}}=\int\frac{1}{r}(ar^2+2Hr-b)^{-1/2}dr, \tag{7}$$

$$J=\int\frac{[d-f(\theta)]d\theta}{\sin\theta\{P^2\sin^2\theta-[d-f(\theta)]^2\}^{1/2}}=\int d\varphi. \tag{8}$$

The problem is now completely solved. An exact solution exists provided the integrals present in (7) and (8) are elementary. Naturally so long as we do not precise our sights the expression exact solution remains ambiguous. We shall therefore restrict ourselves to a complete solution in term of elementary functions or at most elliptic functions. The θ integrations present in (7) and (8) are elementary in only one case and elliptic in three independent cases. All other cases seem unsolvable or reduce to linear combinations of the former.

1. Elementary integrations

Except in the case $g=\mathcal{H}/\sin\theta$, where $\mathbf{B}$ vanishes, there is only one possibility:

$g=\mathcal{H}\cot\theta \rightarrow \mathbf{B}=\mathcal{H}\mathbf{r}/r^3$ (Coulomb magnetic field).

Equations (7) and (8) integrate into elementary functions. For example r and θ are connected by the relation

$$[P^2+(\epsilon\mathcal{H}/\mu_0)^2]^{-1/2}\arcsin\frac{d(\epsilon\mathcal{H}/\mu_0)-[P^2+(\epsilon\mathcal{H}/\mu_0)^2]\cos\theta}{P[P^2+(\epsilon\mathcal{H}/\mu_0)^2-d^2]^{1/2}}=P^{-1}\arcsin\frac{Hr-P^2}{r(H^2+aP^2)^{1/2}}.$$

2. Elliptic integrations

It is known from the theory of elliptic functions[6] that the integral $\int R(z,\mathscr{P}^{1/2})dz$, where the function R is rational in z and in $\mathscr{P}^{1/2}=(Az^4+Bz^3+Cz^2+Dz+E)^{1/2}$ is of elliptic kind. Therefore it is sufficient for our purpose to introduce into (7) and (8) a lot of trigonometrical functions in place of $f(\theta)$ and to retain only those which lead to elliptic integrals when these are made algebraic after a suitable change of variables.

Equations (7) and (8) are automatically rational when one puts $y=\tan\theta/2$; however $\mathscr{P}$ is at most of fourth order if $g=\mathcal{H}$ (=const). Likewise if one puts $u=\cos\theta$ we must restrict ourselves to $g=\mathcal{H}\sin\theta$ or $g=\mathcal{H}\tan\theta$. We have never found other possibilities. Let us now review briefly the three cases:

1st case: $g=\mathcal{H} \rightarrow \mathbf{B}=\mathcal{H}\cot\theta\,\mathbf{r}/r^3$.

One puts $y=\tan\theta/2$. Equations (7) and (8) become

$$I=4\int\frac{y\,dy}{(1+y^2)\mathscr{P}^{1/2}},\quad J=\int\frac{d(1+y^2)-2(\epsilon\mathcal{H}/\mu_0)y}{y\mathscr{P}^{1/2}}dy.$$

$\mathscr{P}$ is defined in accordance with the following values:

$$A=E=-d^2,\qquad B=D=4d(\epsilon\mathcal{H}/\mu_0),\qquad C=4P^2-4(\epsilon\mathcal{H}/\mu_0)^2-2d^2. \tag{9}$$

2nd case: $g=\mathcal{H}\sin\theta \rightarrow \mathbf{B}=2\,\mathcal{H}\cos\theta\,\mathbf{r}/r^3$.

One puts $u=\cos\theta$:

$$I=-\int\frac{du}{\mathscr{P}^{1/2}},\quad J=\int\frac{(\epsilon\mathcal{H}/\mu_0)(1-u^2)-d}{(1-u^2)\mathscr{P}^{1/2}}du,$$

$$A=-(\epsilon\mathcal{H}/\mu_0)^2,\quad B=D=0,\quad C=-P^2+2(\epsilon\mathcal{H}/\mu_0)^2-2d(\epsilon\mathcal{H}/\mu_0),\quad E=P^2-[d-(\epsilon\mathcal{H}/\mu_0)]^2. \tag{10}$$

3rd case: $g=\mathcal{H}\tan\theta \rightarrow \mathbf{B}=\mathcal{H}(2+\tan^2\theta)\mathbf{r}/r^3$.

One puts $u=\cos\theta$:

$$I=-\int\frac{u\,du}{\mathscr{P}^{1/2}},\quad J=\int\frac{(\epsilon\mathcal{H}/\mu_0)(1-u^2)-du}{(1-u^2)\mathscr{P}^{1/2}}du,$$

$$A=-P^2-(\epsilon\mathcal{H}/\mu_0)^2,\quad B=-D=-2d(\epsilon\mathcal{H}/\mu_0),\quad C=P^2-d^2+2(\epsilon\mathcal{H}/\mu_0)^2,\quad E=-(\epsilon\mathcal{H}/\mu_0)^2. \tag{11}$$

To save place we shall omit the complete writing of the solutions in terms of elliptic functions.

Remark: It must be pointed out that conformably to the theory of elliptic functions the elliptic integrals I and J degenerate in elementary integrals if $\mathscr{P}$ has a double root. That is of course possible only if P^2, d, and $\epsilon\mathcal{H}/\mu_0$ (the characteristic parameters of our problem) are suitably connected. For example, in the second case the following condition is needed:

$$P^2=[(\epsilon\mathcal{H}/\mu_0)-d]^2.$$

II. EINSTEIN'S MECHANICS

Einstein's equation is written as:

$$\beta\boldsymbol{\gamma}+(\beta^3/c^2)\mathbf{v}\cdot\boldsymbol{\gamma}\mathbf{v}=(\epsilon/\mu_0)\mathbf{v}\wedge h(\theta)\mathbf{r}/r^3-H\mathbf{r}/r^3. \tag{12}$$

If there was no electric field this equation would be the same as Newton's.[2]

A. The radial integration

The conservation of energy implies

$$\beta-H/(c^2r)=a'\quad(a'=\text{const}). \tag{13}$$

By scalar multiplication of (12) by $\mathbf{r}$ we get

$$\beta\mathbf{r}\cdot\boldsymbol{\gamma}+(\beta^3/c^2)\mathbf{v}\cdot\boldsymbol{\gamma}\mathbf{r}\cdot\mathbf{v}+H/r=0.$$

Hence

$$d(\beta\mathbf{r}\cdot\mathbf{v})/dt=\beta\mathbf{v}^2-H/r.$$

Eliminating β in this equation with the aid of (13) we get

$$\frac{d}{dt}\left((a'c^2r+H)\frac{dr}{dt}\right)=c^4\frac{(a'^2-1)c^2r+a'H}{a'c^2r+H}.$$

The integration of this equation is classical. Putting $w=a'c^2r+H$, one finally arrives at

$$c^2\int dt=\int w[c^2(a'^2-1)w^2+2c^2Hw-b'a'^2]^{-1/2}dw\quad(b'=\text{const}),$$

which defines the radial motion by means of elementary functions without reference to the magnetic field: We refind the radial relativistic Kepler's motion.

B. The angular integration

Let us define $\mathbf{J}=\beta\mathbf{r}\wedge\mathbf{v}$ and let us calculate J^2. Working as in Sec. I B one finds

$$J^2=b'/c^4-H^2/a'^2c^2=\text{const}.$$

From another side

$$d\mathbf{J}/dt=(\epsilon/\mu_0)[h(\theta)/r^3]\mathbf{r}\wedge(\mathbf{v}\wedge\mathbf{r})$$

from which we deduce $J_z = d' - f(\theta)$ ($d' = \text{const}$) as in Sec. I B. Summarizing these two results we arrive on account of (3) to the following integrations which allow us to determinate the angular motion:

$$J^2 = \beta^2 r^4(\dot\theta^2 + \sin^2\theta\dot\varphi^2),$$
$$d' - f(\theta) = \beta r^2 \sin^2\theta\dot\varphi.$$

The angular integrals are

$$\int \frac{\sin\theta\, d\theta}{\{J^2\sin^2\theta - [d' - f(\theta)]^2\}^{1/2}} = a'c^2$$
$$\int \frac{dw}{(w-H)[c^2(a'^2-1)w^2 + 2c^2Hw - ba'^2]^{1/2}},$$
$$\int \frac{[d' - f(\theta)]\,d\theta}{\sin\theta\{J^2\sin^2\theta - [d' - f(\theta)]^2\}^{1/2}} = \int d\varphi.$$

The θ and the φ dependence of these equations are exactly identical with those found in Newton's theory so that the same conclusions remain valuable: one field leads to elementary solutions and three others lead to elliptic integrals. These fields have been presented in Sec. I.

III. SCHRÖDINGER'S MECHANICS

A. Separation of the variables

Schrödinger's equation is written as

$$\Delta\psi + 2(\epsilon/\hbar)i\mathbf{A}\cdot\overrightarrow{\text{grad}}\psi - (\epsilon/\hbar)^2A^2\psi + (2\mu_0/\hbar^2)(E - V)\psi = 0. \tag{14}$$

Using spherical coordinates one establishes with the aid of (4) that

$$\mathbf{A}\cdot\overrightarrow{\text{grad}} = [g/(r^2\sin\theta)]\partial/\partial\varphi,$$
$$A^2 = (g/r)^2.$$

Variables separate in Eq. (14): $\psi = \exp(im\varphi)ⓤR$. The φ equation immediately integrates into classical imaginary exponential form whilst r and θ equations are (m is the integer magnetic quantum number—we shall only consider positive values of m; calculations are analogous when m is negative):

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + \frac{2\mu_0}{\hbar^2}r^2\left(E + \frac{\mu_0 H}{r}\right)R + sR = 0, \tag{15a}$$
$$\frac{d^2ⓤ}{d\theta^2} + \cot\theta\frac{dⓤ}{d\theta} - \left(\frac{m}{\sin\theta} + \epsilon\frac{g}{\hbar}\right)^2ⓤ - sⓤ = 0. \tag{15b}$$

The physically admissible solutions of (15b) only exist for definite values of the constant parameter s. Then Eq. (15a) is analogous to the radial equation in the hydrogen problem (except the values of s). The discrete energy levels are given by

$$E = -(\mu_0^3H^2/2\hbar^2)[n + 1/2 + (1/4 - s)^{1/2}]^{-2}. \tag{16}$$

Therefore there is only one problem: the resolution of (15b) in view of finding the allowed values for s. An exact solution to this problem exists if the θ equation is of a classical type with polynomial solutions (i.e., Hermite, Laguerre, or Jacobi equation). However as we shall see the field of our investigations is then too narrow so that we shall also admit θ equations which after a suitable change of variables belong to the differential equations trilogy we have previously studied.[7]

The reader will understand later our reasons for doing so. In summary we have to introduce various trigonometrical functions in place of g in Eq. (15b) and to see after a suitable change of variables whether or not it is possible to bring the equation into a classical form (Hermite, Laguerre, or Jacobi) or into one of the three nonusual forms[7]:

$$Df'' + (az^2 + bz + c)f' + (d + ez)f = 0, \tag{17}$$

where $D = z$, $z(z-1)$ or $z(1-z)(\alpha - z)$; we shall call these three equations (17a), (17b) and (17c), respectively. In the theory[7] of Eq. (17) the parameters a, b, and c must verify a very simple condition (j, j', and j'' are integers ≥ 0):

For Eq. (17a):

$$c = -j, \tag{18}$$

For Eq. (17b):

$$c = j, \tag{19a}$$

or

$$a + b + c = -j'. \tag{19b}$$

The second relation deduces from the first when z is replaced by $1 - z$ in (17b).

For Eq. (17c):

$$c = -j\alpha, \tag{20a}$$

or

$$a + b + c = -j'(1-\alpha), \tag{20b}$$

or

$$a\alpha^2 + b\alpha + c = -j''\alpha(\alpha - 1). \tag{20c}$$

The second relation (resp. the third) deduces from the first when z is replaced by $1 - z$ (resp. by $\alpha - \alpha z$) in (17c).

Here are the solutions we have found to the problem.

1. Classical equations

Excepting the case $g = \mathcal{K}/\sin\theta$, where $\mathbf{B}$ vanishes, there is only one possibility:

$$g = \mathcal{K}\cot\theta \to \mathbf{B} = \mathcal{K}\mathbf{r}/r^3 \quad \text{(Coulomb field)}.$$

This case leads to Jacobi polynomials if $s = (\epsilon\mathcal{K}/\hbar)^2 - l(l+1)$. We pass over the details since this problem is not new.[8]

2. Nonusual equations of the type (17)

If we put $y = \sin\theta$ in (15b) it may be seen that $g = \mathcal{K}$ ($=$ const) provides a solution to our problem. If we put $u = \cos\theta$ in (15b) it may be seen that $g = \mathcal{K}\sin\theta$ and $g = \mathcal{K}\tan\theta$ are also convenient. We have not succeeded in finding another function g independent of those just mentioned. Let us now review the three cases in greater detail:

1st case: $g = \mathcal{K} \to \mathbf{B} = \mathcal{K}\cot\theta\mathbf{r}/r^3$.

We put $y = \sin\theta$ in (15b)

$$(1 - y^2)\frac{d^2ⓤ}{dy^2} + \frac{1 - 2y^2}{y}\frac{dⓤ}{dy} - \left[\frac{m^2}{y^2} + s + \left(\frac{\epsilon\mathcal{K}}{\hbar}\right)^2 + 2m\frac{\epsilon\mathcal{K}}{\hbar}\frac{1}{y}\right]ⓤ = 0. \tag{21}$$

If $\textcircled{u}=y^{-m}T$ we find

$$y(1-y)(-1-y)T''+[(2-2m)y^2+(2m-1)]T'+\{2m\epsilon\mathcal{H}/\hbar+[s+(\epsilon\mathcal{H}/\hbar)^2+m^2-m]y\}T=0. \tag{22}$$

Equation (22) is of the type (17c). Only condition (20a) is satisfied with $j=2m-1$.

2nd case: $g=\mathcal{H}\sin\theta \rightarrow \mathbf{B}=2\mathcal{H}\cos\theta\mathbf{r}/r^3$.

We put $u=\cos\theta$ in (15b):

$$(1-u^2)\frac{d^2\textcircled{u}}{du^2}-2u\frac{d\textcircled{u}}{du}-\left[\frac{m^2}{1-u^2}+\left(\frac{\epsilon\mathcal{H}}{\hbar}\right)^2(1-u^2)+s+\frac{2m\epsilon\mathcal{H}}{\hbar}\right]\textcircled{u}=0. \tag{23}$$

If $\textcircled{u}=(1-u^2)^{-m/2}\exp[-(\epsilon\mathcal{H}/\hbar)u]T$ and $u=2v-1$ for the sake of convenience:

$$v(v-1)T''+[-4(\epsilon\mathcal{H}/\hbar)v^2+(4\epsilon\mathcal{H}/\hbar-2m+2)v+(m-1)]T'+[(2\epsilon\mathcal{H}/\hbar+s+m^2-m)+4(\epsilon\mathcal{H}/\hbar)(m-1)v]T=0. \tag{24}$$

Equation (24) is of the type (17b). Both conditions (19a) and (19b) are satisfied with $j=j'=m-1$.

3rd case: $g=\mathcal{H}\tan\theta \rightarrow \mathbf{B}=\mathcal{H}(2+\tan^2\theta)\mathbf{r}/r^3$.

We put $u=\cos\theta$ in (15b):

$$(1-u^2)\frac{d^2\textcircled{u}}{du^2}-2u\frac{d\textcircled{u}}{du}-\left[\frac{m^2}{1-u^2}+2m\frac{\epsilon\mathcal{H}}{\hbar}\frac{1}{u}+\left(\frac{\epsilon\mathcal{H}}{\hbar}\right)^2\frac{1-u^2}{u^2}+s\right]\textcircled{u}=0. \tag{25}$$

If $\textcircled{u}=(1-u^2)^{-m/2}u^{\sigma}T$ with $\sigma^2-\sigma-(\epsilon\mathcal{H}/\hbar)^2=0$, we have

$$u(1-u)(-1-u)T''+[2(\sigma+1-m)u^2-2\sigma]T'+\{2m\epsilon\mathcal{H}/\hbar+[-m+s+m^2+2\sigma(1-m)]u\}T=0. \tag{26}$$

Equation (26) is of the type (17c). Both conditions (20b) and (20c) are satisfied with $j'=j''=m-1$.

Digression: In the three cases T obeys an equation like (17). Before going on let us make a digression about the solutions of (17). We recall and extend the results of Ref. 7. Because of the condition of finiteness we shall restrict ourselves to polynomial T functions:

$$T=\sum_0^n \lambda_k z^k.$$

A first polynomial condition is

$$\begin{aligned} e&=-an && \text{for (17a) and (17b)},\\ e&=-n(n+a-1) && \text{for (17c)}. \end{aligned} \tag{27}$$

The recurrence equation which gives λ_k is $R_k\lambda_{k-1}+S_k\lambda_k+T_k\lambda_{k+1}=0$ $(k=0,\ldots,n)$. The coefficients R_k, S_k and T_k are defined in Ref. 7. The last equation is compatible only if

$$\begin{vmatrix} S_0 & T_0 & & \\ R_1 & S_1 & T_1 & \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \\ & R_{n-1} & S_{n-1} & T_{n-1} \\ & & R_n & S_n \end{vmatrix}=0. \tag{28}$$

This determinant equation may be fulfilled in two different ways provided a, b, and c are well-connected [see Eqs. (18), (19), and (20)]. Indeed $T_j=0$ so that two possibilities exist to satisfy (28):

1st possibility:

$$\begin{vmatrix} S_0 & T_0 & & \\ R_1 & S_1 & T_1 & \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \\ & R_{j-1} & S_{j-1} & T_{j-1} \\ & & R_j & S_j \end{vmatrix}=0,$$

2nd possibility: $\lambda_0=\lambda_1=\ldots=\lambda_j=0$,

$$\begin{vmatrix} S_{j+1} & T_{j+1} & & \\ R_{j+1} & S_{j+1} & T_{j+1} & \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \\ & R_{n-1} & S_{n-1} & T_{n-1} \\ & & R_n & S_n \end{vmatrix}=0 \tag{29}$$

so that

$$T=z^{j+1}\times \text{polynomial of degree } n-j-1. \tag{30}$$

We have seen[7] that with the first possibility the solution of (17) appear as linear combinations of Hermite, Laguerre, or Jacobi polynomials. The second possibility is more difficult to deal with because the order of the determinant (29) increases with the order of the polynomial. Yet only the second possibility is interesting for our purpose because of the divergent from of $\textcircled{u}$ in (21), (23), and (25): indeed it will be shown on each particular case that on account of (30) $\textcircled{u}$ is finally regular with the second possibility whilst it would be divergent with the first.

B. Resolution of the differential equations

We only study the three magnetic induction fields mentioned above. As we shall see no physically admissible solution exist unless the parameters characterizing the problem are suitably connected. In particular the parameter $\mathcal{H}$ (which has the dimensions of a magnetic pole) cannot take arbitrary values.

1st case: $\mathbf{B}=\mathcal{H}\cot\theta\mathbf{r}/r^3$.

We must deal with Eq. (22). Because $j=2m-1$ in that case we have on account of (30)

$$T=y^{2m}P^{(\nu)},$$

where $P^{(\nu)}$ denotes a polynomial of degree ν. The first polynomial condition [see (27)] gives the allowed values for s:

$$s=-(\epsilon\mathcal{H}/\hbar)^2-(m+\nu)(m+\nu+1). \tag{31}$$

If $\mathcal{H}=0$ we recover the classical hydrogen values $s=-l(l+1)$ if we set $l=m+\nu$. The second polynomial condition [see the determinant condition (29)] is

$$\begin{vmatrix} S_{2m} & T_{2m} & & \\ R_{2m+1} & S_{2m+1} & T_{2m+1} & \\ \cdot & \cdot & \cdot & \\ & \cdot & \cdot & \cdot \\ & R_{2m+\nu-1} & S_{2m+\nu-1} & T_{2m+\nu-1} \\ & & R_{2m+\nu} & S_{2m+\nu} \end{vmatrix} = 0. \qquad (32)$$

For each value of ν it gives the allowed values for the parameter $\mathcal{K}$. For example, $\nu=1$ (a simple case): the determinant (32) is of order two and the values of R_k, S_k and T_k are given in Ref. 7; one finds

$$\begin{vmatrix} 2m\epsilon\mathcal{K}/\hbar & -2m-1 \\ -2m-2 & 2m\epsilon\mathcal{K}/\hbar \end{vmatrix} = 0,$$

i.e.,

$$\mathcal{K} = (\hbar/2\epsilon)\{[(2m+1)(2m+2)]^{1/2}/m\}.$$

We deduce the value of s in this special case and we find the discrete energy levels numbered by the integer n [see Eq. (16)]:

$$E = -(\mu_0^3H^2/2\hbar^2)(n+1/2+\{1/4+[(m+1)/2m^2](2m^3 + 4m^2+2m+1)\}^{1/2})^{-2}.$$

Each value of ν must be analyzed separately in the same manner. Writing the determinant condition (32) in each cases we can at least, when ν is small, list all the values of $\mathcal{K}$ which allow us to solve completely Schrödinger's equation. Inversely the value of $\mathcal{K}$ being fixed (among the allowed values of course) the corresponding values of m and ν deduce like that of s and finally (16) allows us to construct the energy levels spectrum (with $n=0,1,2,\ldots$). Since m and s are related to the angular momentum proper values we conclude by saying that a given allowed value of $\mathcal{K}$ automatically forbids arbitrary angular momentum states.

2nd case: $\mathbf{B} = 2\mathcal{K}\cos\theta\mathbf{r}/r^3$.

We must deal with Eq. (24). In fact this case is very different from the two others because it does not lead to a discrete energy levels spectrum. Indeed the first polynomial condition [see (27)] can never be satisfied; one must have on account of (30)

$$T = v^m(v-1)^mP^{(\nu)}.$$

So that T would be a polynomial of degree $\nu+2m$; therefore, (27) would need $4(\epsilon\mathcal{K}/\hbar)(m-1) = 4(\epsilon\mathcal{K}/\hbar)(\nu+2m)$, which is impossible. We shall try to see in the final discussion the reasons for that special demeanor.

3rd case: $\mathbf{B} = \mathcal{K}(2+\tan^2\theta)\mathbf{r}/r^3$.

We must deal with Eq. (26). Because $j'=j''=m-1$ in that case we have on account of (30)

$$T = (1-u^2)^mP^{(\nu)}. \qquad (33)$$

The first polynomial condition [see (27)] gives the allowed values for s:

$$s = -(\nu+2\sigma+m)(\nu+m+1).$$

If $\mathcal{K}=0$ we recover the classical value $s=-l(l+1)$ by setting $l=\nu+m$. To obtain the polynomial determinant condition we must introduce (33) into (26). Just like in the first case each value of ν must be analyzed separately. The determinant condition gives in each cases the allowed $\mathcal{K}$ values. Let us treat again the special case $\nu=1$; the determinant is of order two:

$$\begin{vmatrix} 2m\epsilon\mathcal{K}/\hbar & -2\sigma \\ -2m-2\sigma-2 & 2m\epsilon\mathcal{K}/\hbar \end{vmatrix} = 0.$$

Remembering the definition $\sigma[\sigma^2-\sigma-(\epsilon\mathcal{K}/\hbar)^2=0]$, one finds the allowed $\mathcal{K}$ values:

$$\mathcal{K} = (\hbar/\epsilon)\{[m+2)(m^2+m+1)]^{1/2}/(m^2-1)\}.$$

The corresponding s values follow and so do the energy levels

$$E = -(\mu_0^3H^2/2\hbar^2)(n+1/2+\{1/4+[(m+2)/(m^2-1)](m^3 + 3m^2+m+1)\}^{1/2})^{-2}.$$

Remark: Before concluding this section it is interesting to recall what has been done. Schrödinger's equation seems completely solvable in only one case. In three other cases it is conditionally solvable: $\epsilon\mathcal{K}/\hbar$ must be correctly connected with the angular momentum quantum numbers so that $\mathcal{K}$ is irrational in unit $\hbar/\epsilon$.

IV. DIRAC'S MECHANICS

In this section we deal with very complicated equations. Our first care must be the separation of the spherical variables in Dirac's equation. We have shown elsewhere that this operation is performed in the easiest way by using the quaternionic formalism.[7,9] To save place here the reader is referred to Ref. 7, Sec. IB to discover the details of the quaternionic procedure: it is shown that the φ equation simply leads to trigonometrical functions and that the radial equation is analogous to that of the hydrogen atom problem except that $(J+1/2)^2$ is replaced by the parameter $-\lambda^2$ to be determined. So we deduce that the energy levels are given by the well-known hydrogen formula

$$E = \mu_0c^2(1+(\mu_0H/c\hbar)^2\{n+[-\lambda^2-(\mu_0H/c\hbar)^2]^{1/2}\}^{-2})^{-1/2}. \qquad (34)$$

The θ equation alone depends upon the presence of the magnetic potential. Therefore we have only to solve the following coupled system:

$$\frac{dT_1}{d\theta} = -\lambda T_4 - \frac{(2m+1)+\cos\theta}{2\sin\theta}T_1 - (\epsilon g/\hbar)T_1,$$
$$\frac{dT_4}{d\theta} = -\lambda T_1 + \frac{(2m+1)-\cos\theta}{2\sin\theta}T_4 + (\epsilon g/\hbar)T_4. \qquad (35)$$

This system is exactly solvable by means of elementary functions in the case $g=\mathcal{K}\cot\theta$, i.e., for the field of a magnetic pole. We pass over the details since the problem has been investigated by Harish-Chandra.[8] It is conditionally solvable in three cases:

1st case: $g=\mathcal{K} \rightarrow \mathbf{B} = \mathcal{K}\cot\theta\mathbf{r}/r^3$.

We make the substitution $z=\exp(i\theta)$ in (35) and we decouple the system to obtain two second-order rational equations. We shall only deal with the T_1 equation since when T_1 is known, T_4 deduces by (35); it is written with arbitrary g function ($g'=dg/d\theta$):

$$z^2\frac{d^2T_1}{dz^2} + \frac{2z^3}{z^2-1}\frac{dT_1}{dz} + \Big[(\lambda^2+1/2) - \frac{4m^2z^2}{(z^2-1)^2} - \frac{2mz}{(z-1)^2}$$

$$-\left(\frac{1}{4}\right)\left(\frac{z+1}{z-1}\right)^2+2i(2m+1)\ \frac{\epsilon}{\hbar}\ g\frac{z}{z^2-1}+\left(\frac{\epsilon g}{\hbar}\right)^2 - \frac{\epsilon g'}{\hbar}\Bigg]\ T_1=0.$$

This equation is too general, since we only deal with $g=\mathcal{H}$ (=const) so that $g'=0$. Let us set

$$T_1=(z-1)^{-m-1}(z+1)^{-m}z^{\sigma}V_1,$$

where $\sigma^2-\sigma+\lambda^2+(\epsilon\mathcal{H}/\hbar)^2+1/4=0$. We find

$$z(1-z)(-1-z)V_1''+[(2\sigma-4m)z^2-2z-2\sigma]V_1' +\{[(2m+1)(1+2i\epsilon\mathcal{H}/\hbar)-2\sigma] +(4m^2+2m-4m\sigma)z\}V_1=0. \tag{36}$$

Equation (36) is of the type (17c). Both conditions (20b) and (20c) are satisfied with $j'=2m+1$ and $j''=2m-1$. Therefore, we have on account of (30)

$$V_1=(z-1)^{2m+2}(z+1)^{2m}P^{(\nu)},$$

where $P^{(\nu)}$ is a polynomial of degree ν. The first polynomial condition [see (27)] gives the allowed values for λ:

$$4m^2+2m-4m\sigma=-(\nu+4m+2)(\nu+2\sigma+1), \tag{37}$$

which reduces after simplifications to

$$2\sigma+2m+\nu+1=0. \tag{38}$$

If $\mathcal{H}=0$ we recover the classical hydrogen value $\lambda^2=-(J+1/2)^2$ if we set $J=1/2(\nu+2m+1)$ with ν even. When ν is odd no classical equivalent exists. To obtain the polynomial determinant condition we must introduce (37) in (36). Each value of ν must be analyzed separately. Let us treat the case $\nu=1$; the determinant is of order $\nu+1=2$:

$$\begin{vmatrix} 2i(2m+1)(\epsilon\mathcal{H}/\hbar)-1 & 2m+2 \\ -2m-2 & 2i(2m+1)(\epsilon\mathcal{H}/\hbar)+1 \end{vmatrix}=0,$$

i.e.,

$$\mathcal{H}=\frac{\hbar}{2\epsilon}\left(\frac{2m+3}{2m+1}\right)^{1/2}.$$

The fact that these values are not identical with those found in the nonrelativistic theory may not surprise since in Dirac's theory the charged particle automatically carries a magnetic moment.

We deduce from (38) and the definition of σ the value of λ^2 in this special case ($\nu=1$):

$$\lambda^2=-[(m+1)^2(2m+3)]/(2m+1).$$

Finally, on account of (34) the energy levels are

$$E=\mu_0c^2[1+(\mu_0H/c\hbar)^2(n+\{[(m+1)^2(2m+3)/(2m+1)] -(\mu_0H/c\hbar)^2\}^{1/2})^{-2}]^{-1/2}.$$

Let us remark in passing that the presence of the magnetic field allows us to consider a strong Coulomb field. In the hydrogen theory one is limited by the condition $\mu_0H/c\hbar\leqslant 1$ equivalent to $Z<137$ if Z is the atomic number but here the limitation is not so strict: $\mu_0H/c\hbar\leqslant 3$ equivalent to $Z<411$.

2nd case: $g=\mathcal{H}\sin\theta\rightarrow \mathbf{B}=2\mathcal{H}\cos\theta\mathbf{r}/r^3$.

We make the substitution $u=\cos^2\theta/2$ in (35) and we decouple the system. We again only retain the T_1 equation; for arbitrary g we have as follows:

$$u(1-u)\frac{d^2T_1}{du^2}-(2u-1)\frac{dT_1}{du}-\left[(\lambda^2+1/2)+\frac{(m+u)^2}{4u(1-u)} + \frac{\epsilon g}{\hbar}\ \frac{2m+1}{\sin\theta}+\left(\frac{\epsilon g}{\hbar}\right)^2-\frac{\epsilon g'}{\hbar}\right]T_1=0. \tag{39}$$

Here we deal with the special case $g=\mathcal{H}\sin\theta$ so that $g'=\mathcal{H}\cos\theta$. We set

$$T_1=u^{-m/2}(1-u)^{-(m+1)/2}\exp[-2(\epsilon\mathcal{H}/\hbar)u]V_1,$$

where

$$u(u-1)V_1''+\{-4(\epsilon\mathcal{H}/\hbar)u^2+[4(\epsilon\mathcal{H}/\hbar)-2m+1]u+(m-1)\}V_1' +\{(\lambda^2+m^2+4\epsilon\mathcal{H}/\hbar)-4(m-1)(\epsilon\mathcal{H}/\hbar)u\}V_1=0.$$

This equation is of the type (17b). Both conditions (19a) and (19b) are satisfied with $j=m-1$ and $j'=m$. However it is impossible to set here

$$V_1=u^m(1-u)^{m+1}P^{(\nu)},$$

where $P^{(\nu)}$ would be a polynomial of degree ν. Indeed the first polynomial condition [see (27)] can never be satisfied. Therefore no quantization of energy exists. The situation was similar in Schrödinger's theory.

3rd case: $g=\mathcal{H}\tan\theta\rightarrow \mathbf{B}=\mathcal{H}(2+\tan^2\theta)\mathbf{r}/r^3$.

We again make the substitution $u=\cos^2\theta/2$ in (35) so that we can start with (39) where $g'=\mathcal{H}/\cos^2\theta$. Let us put in (39):

$$T_1=(2u-1)^{\epsilon\mathcal{H}/\hbar}u^{-m/2}(1-u)^{-(m+1)/2}V_1.$$

One deduces

$$u(1-u)(1/2-u)V_1''+[(2\epsilon\mathcal{H}/\hbar-2m+1)u^2+(2m-3/2 -2\epsilon\mathcal{H}/\hbar)u-(m-1)/2]V_1'+\{-(\lambda^2+m^2)/2 +[\lambda^2+m^2+2(m+1)\epsilon\mathcal{H}/\hbar]u\}\ V_1=0. \tag{40}$$

Equation (40) is of the type (17c). Both conditions (20a) and (20b) are satisfied with $j=m-1$ and $j'=m$. Therefore, we have on account of (30)

$$V_1=u^m(1-u)^{m+1}P^{(\nu)}, \tag{41}$$

where $P^{(\nu)}$ is a polynomial of degree ν.

The first polynomial condition [see (27)] gives the allowed values for λ^2:

$$\lambda^2+m^2+2(m+1)\epsilon\mathcal{H}/\hbar=-(\nu+2m+1)(\nu+2\epsilon\mathcal{H}/\hbar+1). \tag{42}$$

If $\mathcal{H}=0$ we recover $\lambda^2=-(J+1/2)^2$ as in the hydrogen theory provided one sets $J=\nu+m+1/2$.

To obtain the polynomial determinant condition we must introduce (41) into (40). Each value of ν must be analyzed separately. Let us treat again the case $\nu=1$; the determinant is of order two:

$$\begin{vmatrix} m\epsilon\mathcal{H}/\hbar+m+3\epsilon\mathcal{H}/\hbar+3/2 & (m+1)/2 \\ -2m-3-2\epsilon\mathcal{H}/\hbar & (m+1)(\epsilon\mathcal{H}/\hbar-1) \end{vmatrix}=0,$$

where account has been taken of (42).

We deduce

$$\mathcal{H}=(\hbar/2\epsilon)(m+3)^{-1}.$$

These are the $\mathcal{H}$-allowed values in the special case $\nu = 1$. From (42) we deduce the corresponding values of λ^2 and finally the energy levels by (34):

$$E = \mu_0 c^2 (1 + (\mu_0 H/c\hbar)^2 \{n + [(m^3 + 7m^2 + 19m + 15)/(m+3) - (\mu_0 H/c\hbar)^2]^{1/2}\}^{-2})^{-1/2}.$$

We remark like in the first case that the presence of the special magnetic field here considered allows to deal with strong Coulomb fields. The limitation is $\mu_0 H/c\hbar \leqslant 5$ equivalent to $Z < 685$ in the hydrogen theory.

V. DISCUSSION

It is time to compare the solutions obtained in the four fundamental mechanics for the motion of a charged particle in a radial magnetic field of the type (1). The fact that there are analogies between the four treatments is not of course surprising. Let us emphasize them. In nonquantum mechanics: there is only one field (1) which leads to elementary integrations. Three other fields lead to elliptic integrals. All others seem to be more complicated (we shall say unsolvable). In quantum mechanics there is only one field which leads to classical differential equations. Three other fields lead to non-usual differential equations of the type (17). All other fields seem to be unsolvable in that frame. What is very remarkable is that these fields are the same in the four mechanics. That analogy is purely formal and may be pursued in the following way: the final θ integration (resp. the final θ-differential equation) needs a suitable change of variables. In the four treatments this change of variables is analogous when one considers the cases $g_2 = \mathcal{H} \sin\theta$ and $g_3 = \mathcal{H} \tan\theta$ but is different for $g_1 = \mathcal{H}$. Let us now look at more physical analogies.

When one deals with the solutions of the quantum equations in the cases g_1, g_2, and g_3 one may say:

(1) For what regards the cases g_1 and g_3 the problem is conditionally entirely solvable (with discrete energy levels): the parameter $\mathcal{H}$ entering into the definition of g may only take well-chosen values. When this value is fixed the angular momentum quantum numbers m and ν are also fixed at definite values. Such a demeanor is not entirely new in Schrödinger's theory: it is known that definite central electric potentials, like for example $V = V_0 \exp(-r/d)$ or $V = V_0 \tanh^2(r/d)$, lead to solvable equations if and only if the angular momentum quantum number $l = 0$.[10]

(2) The case g_2 is very different since the energy spectrum is not discrete on account of the absence of polynomial solutions.

In nonquantum mechanics the situation is quite analogous:

(1) We have already pointed out in remark (1) that when $\epsilon\mathcal{H}/\mu_0$, P and P_z are suitably connected the elliptic integrals may degenerate into elementary ones. So both integrals I and J [see (7) and (8)] become of the type $\int R[z, (az^2 + bz + c)^{1/2}]dz$, where R denotes a rational function. It is known that following the sign of a and c the dependence upon z of that integral may be inverse trigonometrical or logarithmical. It might be shown with the aid of (9), (10), and (11) that in the cases of g_1 and g_3 the solution is not logarithmical so that the motion is a stable orbit (i.e., $r_{\min} < r < r_{\max}$).

(2) Quite the contrary in the case of g_2 the integral is always logarithmical so that the motion is not stable: the particle falls on the center. As Gupta[11] pointed out it does not correspond discrete energy levels to such a spiral orbit in the equivalent quantum problem.

We may conclude be saying that the quadruple treatment of a same problem in four different mechanics exhibits expected physical analogies and also formal analogies which are sometimes of a strange kind. A stable motion is possible in a Coulomb magnetic field but also in the fields $\mathbf{B} = \mathcal{H} \cot\theta \mathbf{r}/r^3$ or $\mathbf{B} = \mathcal{H}(2 + \tan^2\theta)\mathbf{r}/r^3$ provided the value of $\mathcal{H}$ is allowed.

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