# Some comments on a recent paper of Glasser

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# Some comments on a recent paper of Glasser [J. Math. Phys. 14, 409 (1973)]

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The calculations which lead to results recently established by Glasser are simplified.

In a recent issue of the journal [J. Math. Phys. 14, 409 (1973)], Glasser published a very interesting paper dealing with the possibility of evaluating exactly certain double series. The final results are expressed by him in terms of Riemann zeta and beta functions; they are listed in Table I of his paper. The aim of this short note simply lies in the two following point:

(a) An error has been committed in the evaluation of $S_2$; we correct it.
(b) it is possible to simplify greatly a part of the calculations, especially those which lead to the value of $S_5$.

Let us first recall the notations used: $m$ and $n$ run over all positive integers; $p$ and $q$ run over all positive even integers; and $k$ and $l$ run over all positive odd integers. We wish to evaluate the following sums:

$$S_1 = \sum_{m,n} (m^2 + n^2)^{-s}, \qquad S_2 = \sum_{m,n} (-1)^{m+n}(m^2 + n^2)^{-s},$$

$$S_3 = \sum_{m,n} (-1)^{m-1} (m^2 + n^2)^{-s}, \qquad S_4 = \sum_{k,l} (k^2 + l^2)^{-s},$$

$$S_5 = \sum_{m,p} (m^2 + p^2)^{-s}, \qquad S_6 = \sum_{k,p} (k^2 + p^2)^{-s},$$

$$S_7 = \sum_{m,k} (m^2 + k^2)^{-s}.$$

As we shall see later it is only necessary to evaluate two of these sums since all the others will follow through elementary arithmetical deductions. It is possible to simplify the procedure indicated by Glasser in the following way. To evaluate $S_1$ and $S_3$ we start with Jacobi's identities ($|q| < 1$):

$$\sum_0^\infty \sum_0^\infty (-1)^r q^{(2r+1)(t+1)} = \sum_0^\infty \sum_1^\infty q^{m^2+n^2},$$

$$\sum_0^\infty \sum_0^\infty (-1)^{r+t} q^{(4r+2)(t+1)} = \sum_1^\infty \sum_1^\infty (-1)^{m+1} q^{m^2+n^2} - \sum_1^\infty q^{4n^2}.$$

The reader who is not familiar with the theory of the theta functions can verify these identities by equating in the two members the terms of equal power in $q$. Put $q = e^{-x}$, multiply the two members by $x^{s-1}$ and integrate both sides between $x = 0$ and $x = \infty$. The results announced by Glasser are immediate. The evaluation of the other sums can, of course, be performed through the same procedure; but there is a simpler way. It is evident that the set of positive integers can be split into two subsets: the set of positive odd integers and the set of positive even integers. Our conclusion is that one has: $2S_7 = S_1 + S_3$ and $2S_5 = S_1 - S_3$ through a simple arithmetical device.

The reader will prove without difficulties that

$$S_2 = (2^{2-2s} - 1) S_1 + 2S_3,$$

$$S_4 = 2^{-2s} S_1 + S_3,$$

$$2S_6 = (1 - 2^{1-2s}) S_1 - S_3.$$

Thus we have proved that all the sums $S_1 \cdots S_7$ are deduced linearly from $S_1$ and $S_3$. Finally, one has the following table (with the corrected value of $S_2$):

$$S_1 = \zeta(s)\beta(s) - \zeta(2s),$$

$$S_2 = (1 - 2^{1-2s})\zeta(2s) - (1 - 2^{1-s})\zeta(s)\beta(s),$$

$$S_3 = 2^{-s}[2^{-s}\zeta(2s) + (1 - 2^{1-s})\zeta(s)\beta(s)],$$

$$S_4 = 2^{-s}(1 - 2^{-s})\zeta(s)\beta(s),$$

$$S_5 = \tfrac{1}{2}(1 - 2^{-s} + 2^{1-2s})\zeta(s)\beta(s) - \tfrac{1}{2}(1 + 2^{-2s})\zeta(2s),$$

$$S_6 = \tfrac{1}{2}(1 - 2^{-s})\zeta(s)\beta(s) - \tfrac{1}{2}(1 - 2^{-2s})\zeta(2s),$$

$$S_7 = \tfrac{1}{2}(1 + 2^{-s} - 2^{1-2s})\zeta(s)\beta(s) - \tfrac{1}{2}(1 - 2^{-2s})\zeta(2s).$$

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# Erratum: Modified Lippmann–Schwinger equations for two-body scattering theory with long-range interactions [J. Math. Phys. 14, 1398 (1973)]

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The definition of $D$ in (4.5) should read

$$(D\chi_{[a,b)})(\lambda) = \left|\lambda^{1/2}\left(1 - \cos\frac{q_0}{\sqrt{\lambda}} \log\left|\frac{\lambda - b}{\lambda - a}\right|\right)\right|^{1/2}.$$

On the right-hand side of the inequality (4.8) there should be $x$ instead of $x^2$. With these corrections Lemma 4.1 still shows that $\| Z_{\mp}^{\Delta} \chi_{[a,b)} \| \le \text{const} \| \chi_{[a,b)} \|$ for $[a, b) \subset \Delta$, but it is not sufficient for establishing boundedness for the general case when $Z_{\mp}^{\Delta}$ is applied to an arbitrary element of the form (4.6). In order to prove boundedness in this case, improved estimates taking into account the highly oscillatory behaviour of the kernel of $Z_{\mp\epsilon}^{\Delta}$ are required.