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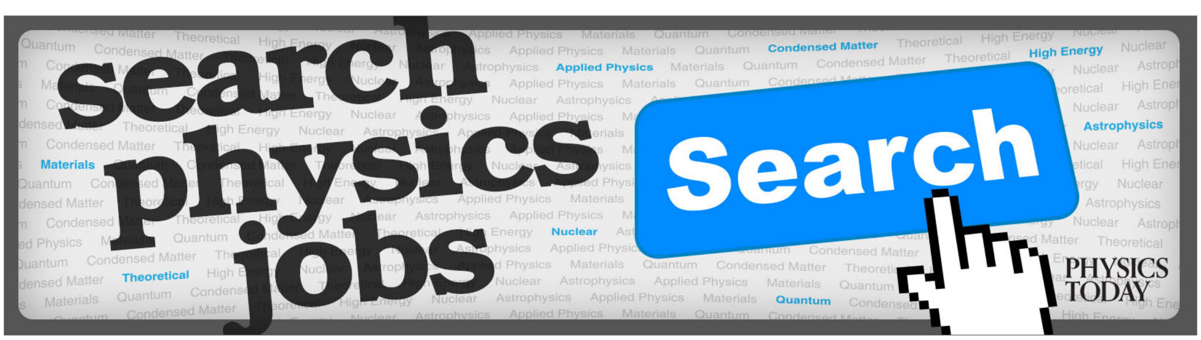
Citation: Journal of Mathematical Physics **14**, 1320 (1973); doi: 10.1063/1.1666184
View online: http://dx.doi.org/10.1063/1.1666184
View Table of Contents: http://aip.scitation.org/toc/jmp/14/10
Published by the American Institute of Physics



Exact motion in noncentral electric fields

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We study the problem of the motion of a charged particle in noncentral potentials of the type $f(\theta)/r^2 + V(r)$. Newton's and Schrödinger's mechanics are considered. Exact solutions exist if $V(r) = -H/r$ or Kr^2 (i.e., Coulomb or harmonic oscillator potentials) while $f(\theta)$ may have at least three different expressions as a function of θ if the problem is three-dimensional and seven expressions if it is two-dimensional. The classical trajectories are computed and the energy levels in the corresponding quantum problem are given. Analogies between the two treatments are discussed.

For a presumed complete bibliography about the problem of finding exact solutions to the equations of motion in the presence of unusual types of potentials see Refs. 1-10. Central magnetic fields were treated in a previous paper.[10] We now turn to noncentral electric potentials of the type

$$J = (\mu/\epsilon)[f(\theta)/r^2 + V(r)], \tag{1}$$

where μ is the mass and ϵ the charge of the particle, (r, θ, φ) its spherical coordinates. V represents its velocity, γ its acceleration. i equals $\sqrt{-1}$. We shall perform the calculations both for Newton's and for Schrödinger's mechanics.

I. NEWTON'S MECHANICS

Newton's equation can be written in the form

$$\gamma = -\,\mathbf{grad}\,[f(\theta)/r^2 + V(r)]$$

or in more detail

$$\gamma = -(1/r)V'(r)\mathbf{r} + 2f(\theta)\,\mathbf{r}/r^4 - (1/r^4)f'(\theta) \times [xz(x^2+y^2)^{-1/2}, yz(x^2+y^2)^{-1/2}, -(x^2+y^2)^{1/2}]. \tag{2}$$

A. The radial integration

The conservation of energy implies

$$v^2 + 2f(\theta)/r^2 + 2V(r) = a \quad (= \text{const}). \tag{3}$$

By scalar multiplication of (2) by $\mathbf{r}$ we obtain:

$$\mathbf{r}\cdot\gamma = -rV'(r) + 2f(\theta)/r^2.$$

From this equation we deduce

$$d(\mathbf{r}\cdot\mathbf{v})/dt = \mathbf{r}\cdot\gamma + v^2 = a - rV'(r) - 2V(r).$$

Remembering that $2\mathbf{r}\cdot\mathbf{v} = dr^2/dt$, we have after a classical integration

$$\mathbf{r}\cdot\mathbf{v} = (ar^2 - 2r^2V(r) - b)^{1/2}. \tag{4}$$

Finally,

$$\int dt = \int r(ar^2 - 2r^2V - b)^{-1/2}\,dr = F(r). \tag{5}$$

It is very remarkable that the radial motion is independent of the noncentral term in the potential (1). Equation (5) is exactly integrable by means of circular functions in the two classical cases:

$$V = V_1 = -H/r \qquad \text{(Coulomb potential)},$$

$$V = V_2 = Kr^2 \qquad \text{(harmonic oscillator)}.$$

In what follows we shall always restrict ourselves to these two possibilities and concentrate on the noncentral term in (1). It must be pointed out that $a > 0$ if $V = Kr^2$, but that $a < 0$ if $V = -H/r$ (for bound states).

B. Angular integrations

Let us define $\mathbf{P} = \mathbf{r} \times \mathbf{v}$; its modulus squared P^2 can be written as

$$P^2 = r^2v^2 - (\mathbf{r}\cdot\mathbf{v})^2.$$

Using (3) and (4) one finds

$$P^2 = b - 2f(\theta).$$

Although the modulus of the angular momentum is not a constant of the motion, the expression $P^2 + 2f(\theta)$ is conserved. On the other hand, in spherical coordinates, P^2 equals $r^4(\dot\theta^2 + \sin^2\theta\dot\varphi^2)$ where the point denotes time differentiation, and

$$r^4(\dot\theta^2 + \sin^2\theta\dot\varphi^2) = b - 2f(\theta). \tag{6}$$

By vector multiplication of (2) by $\mathbf{r}$ one finds

$$d\mathbf{P}/dt = -[f'(\theta)/r^2][-y(x^2+y^2)^{-1/2}, x(x^2+y^2)^{-1/2}, 0].$$

One concludes that $dP_z/dt = 0$, and after integration

$$P_z = r^2 \sin^2\theta\varphi = d \qquad (d = \text{const}). \tag{7}$$

Equations (6) and (7) allow us to find the two last integrations needed for the complete solution of this problem:

$$I = \int \frac{\sin\theta\, d\theta}{\sqrt{[b - 2f(\theta)]\sin^2\theta - d^2}} = \int \frac{1}{r}(ar^2 - 2r^2V - b)^{-1/2}dr, \tag{8}$$

$$J = d\int \frac{d\theta}{\sin\theta\sqrt{[b - 2f(\theta)]\sin^2\theta - d^2}} = \int d\varphi. \tag{9}$$

The problem is now solved. An exact solution exists provided the integrals present in (8) and (9) are elementary. The expression "exact solution" has been defined in our previous paper[10] as solutions expressible in terms of circular or at most elliptic functions.

Remark 1: Equation (7) indicates that P_z is a constant of the motion. If we can choose the Oxz plane so that it contains the initial vectors $\mathbf{r}(t=0)$ and $\mathbf{v}(t=0)$ $\mathbf{P}$ is directed along the y axis, so that $P_z = d = 0$. The trajectory is entirely contained in the Oxz plane: r and θ are the polar coordinates in the Oxz plane. Equation (9) vanishes and (8) determines the polar equation of the trajectory.

Remark 2: In connection with Remark 1 it must be pointed out that classical mechanics makes no distinction between the motion in the three-dimensional potential (1) (where $r^2 = x^2 + y^2 + z^2$) and the motion in the two-dimensional potential (1) (where $r^2 = x^2 + y^2$) provided $d = 0$ in the first case and $v_z = 0$ in the second case. In both cases the trajectory is located in a plane. It can happen (see examples below) that quadratures (8) and (9) may be exactly performed when $d = 0$ but not $d \neq 0$: then the exact solution for the three-dimensional problem only exists with suitable initial conditions (see Remark 1) without equivalent in the quantum formalism. With respect to the Schrödinger equation, an exact solution is to be expected only for the two-dimensional problem (i.e., in cylindrical coordinates).

1. *Elementary integrations*

In Remark 2 it was shown that if a Newtonian problem involving a potential of the type (1) is soluble in two dimensions it is also soluble in three dimensions provided suitable initial conditions are imposed. In fact, it suffices to choose adequately the orientation of the axis of reference. In view of future convenience in the comparison between the classical and quantum treatments of a same problem we must however distinguish potentials which lead to elementary quadratures for arbitrary d from those which need $d = 0$. It is not difficult to see that the following functions $f(\theta)$ satisfy the required condition.

Elementary integrations for arbitrary d-values:

(a) $$f(\theta) = (\hbar^2/2\mu^2)(\alpha \cos^2\theta + \beta \cos\theta + \gamma) \sin^{-2}\theta, \tag{10}$$

(b) $$f(\theta) = (\hbar^2/2\mu^2)(\alpha \cos^4\theta + \beta \cos^2\theta + \gamma) \sin^{-2}\theta \cos^{-2}\theta, \tag{11}$$

(c) $$f(\theta) = (\hbar^2/2\mu^2)(\alpha \cot^2\theta + \beta \cot\theta + \gamma). \tag{12}$$

Elementary integrations when $d = 0$ only:

(d) $$f(\theta) = (\hbar^2/2\mu^2)(\alpha \sin^2\theta + \beta \sin\theta + \gamma) \cos^{-2}\theta, \tag{13}$$

(e) $$f(\theta) = (\hbar^2/2\mu^2)(\alpha \tan^2\theta/2 + \beta \tan\theta/2 + \gamma), \tag{14}$$

(f) $$f(\theta) = (\hbar^2/2\mu^2)(\alpha \cot^2\theta/2 + \beta \cot\theta/2 + \gamma), \tag{15}$$

(g) $$f(\theta) = (\hbar^2/2\mu^2)(\alpha \tan^2\theta + \beta \tan\theta + \gamma). \tag{16}$$

The parameters α, β, and γ have arbitrary constant values (apart from conditions specified below). The factor $\hbar^2/2\mu^2$ is introduced in view of future convenience. We shall solve in detail the problem involving the first of these potentials. For the others we shall restrict ourselves to the equation of the trajectory. We shall also investigate the condition under which the motion is stable, i.e.,

$$0 < r_{\min} < r < r_{\max}.$$

(a) Except for the trivial case $\alpha = \beta = \gamma = 0$, only one special case of (10) seems to have been investigated in the literature:[1] $\beta = 0$, $\alpha = -\gamma$ so that the total potential (1) remains central. The study of the motion in the electrical potential

$$J = (\mu/\epsilon)\,[(\hbar^2/2\mu^2) \times (\alpha \cos^2\theta + \beta \cos\theta + \gamma)/(r^2 \sin^2\theta) + V(r)], \tag{17}$$

where $V(r) = -H/r$ or Kr^2, seems to be new. We only integrate the case $d = 0$ for the sake of brevity. With the new variable $u = \cos\theta$ Eq. (8) becomes

$$-\int [-(b + \alpha\hbar^2/\mu^2)u^2 - (\beta\hbar^2/\mu^2)u + (b - \hbar^2\gamma/\mu^2)]^{-1/2} \times du = \int (ar^2 - 2r^2 V - b)^{-1/2} (1/r)\, dr.$$

We shall see in the final discussion that these quadratures lead to circular functions only if α, β, and γ obey the conditions

$$\alpha + \beta + \gamma \geqslant 0, \qquad \alpha - \beta + \gamma \geqslant 0. \tag{18}$$

If not, the solutions are logarithmic so that the motion is not stable. We shall therefore impose (18). Finally, for $V = V_1 = -H/r$ one obtains the polar equation of the trajectory:

$$(b + \alpha\hbar^2/\mu^2)^{-1/2} \arcsin \frac{2(b + \alpha\hbar^2/\mu^2) \cos\theta + \beta\hbar^2/\mu^2}{[\beta^2\hbar^4/\mu^4 + 4(b + \alpha\hbar^2/\mu^2)(b - \gamma\hbar^2/\mu^2)]^{1/2}} = b^{-1/2} \arcsin \frac{b - Hr}{r(H^2 + ab)^{1/2}} + C^t. \tag{19}$$

The trajectory is a rosette contained between two circles so that

$$b(H + \sqrt{H^2 + ab})^{-1} < r < b(H - \sqrt{H^2 + ab})^{-1}.$$

It is not difficult to see that the trajectory is closed if

$$b^{-1/2}(b + \alpha\hbar^2/\mu^2)^{1/2} = m/n, \text{ a rational number.}$$

When $V = V_2 = Kr^2$, the trajectory is found to be

$$(b + \alpha\hbar^2/\mu^2)^{-1/2} \arcsin \frac{2(b + \alpha\hbar^2/\mu^2) \cos\theta + \beta\hbar^2/\mu^2}{[\beta^2\hbar^4/\mu^4 + 4(b + \alpha\hbar^2/\mu^2)(b - \gamma\hbar^2/\mu^2)]^{1/2}} = (1/2)\, b^{-1/2} \arcsin \frac{2b - ar^2}{r^2(a^2 - 8bK)^{1/2}} + C^t$$

which again presents the aspect of a rosette so that

$$(2b)^{1/2} (a + \sqrt{a^2 - 8bK})^{-1/2} < r < (2b)^{1/2} \times (a - \sqrt{a^2 - 8bK})^{-1/2}.$$

It is closed if the above condition is fulfilled.

(b) We now study the motion in the electrical potential

$$J = (\mu/\epsilon)[(\hbar^2/2\mu^2) (\alpha \cos^4\theta + \beta \cos^2\theta + \gamma)/(r^2 \sin^2\theta \cos^2\theta) + V(r)]. \tag{20}$$

Equation (8) can immediately be integrated provided $\cos^2\theta$ is taken as new variable. The equation of the trajectory is (take $V = V_1$ for example)

$$(1/2)(b + \alpha\hbar^2/\mu^2)^{-1/2} \arcsin \frac{2(b + \alpha\hbar^2/\mu^2) \cos^2\theta - (b - \beta\hbar^2/\mu^2)}{[(b - \beta\hbar^2/\mu^2)^2 - 4(b + \alpha\hbar^2/\mu^2)\hbar^2\gamma/\mu^2]^{1/2}} = b^{-1/2} \arcsin \frac{b - Hr}{r(H^2 + ab)^{1/2}} + C^t.$$

The following conditions are needed for a stable motion:

$$\alpha + \beta + \gamma \geqslant 0, \quad \gamma \geqslant 0. \tag{21}$$

(c) We study the motion in the electric potential

$$J = (\mu/\epsilon)\,[(\hbar^2/2\mu^2)(\alpha \cot^2\theta + \beta \cot\theta + \gamma) + V(r)]. \tag{22}$$

Equation (8) can be integrated provided the new (complex!) variable $e^{2i\theta}$ is taken. The equation of the trajectory is ($V = V_1$):

$$(i/2)[\hbar^2(\gamma - \alpha + i\beta)/\mu^2 - b]^{-1/2} \arcsin \frac{[\hbar^2(\beta + 2i\alpha)/\mu^2]\cot\theta + [\hbar^2(2\gamma + i\beta)/\mu^2] - 2b}{(\cot\theta - i)[(\hbar^4\beta^2/\mu^4) + 4(\hbar^2\alpha/\mu^2)(b - \hbar^2\gamma/\mu^2)]^{1/2}} + \text{c.c.} = b^{-1/2} \arcsin \frac{b - Hr}{r(H^2 + ab)^{1/2}} + \text{const.} \tag{23}$$

No restrictions on α, β, and γ have to be imposed.

In the cases (a), (b), and (c) both integrals (8) and (9) lead to elementary functions for arbitrary d-values. It is only for the sake of brevity that we put $d = 0$ in the equation of the trajectory. In the following problems (d), (e), (f), and (g) the condition $d = 0$ is needed.

(d) We study the motion in the electric potential

$$J = (\mu/\epsilon)[(\hbar^2/2\mu^2)(\alpha \sin^2\theta + \beta \sin\theta + \gamma)/(r^2\cos^2\theta) + V(r)]. \tag{24}$$

Here it is necessary to choose the orientation of the axis of reference so that $d = 0$ in (8) (see Remarks 1 and 2) if we want elementary quadratures. The equation of the trajectory is obtained by simply replacing θ by $\pi/2 - \theta$ in (19). Indeed potential (13) deduces from (10) in that way. However, (8) indicates that the sign of the first member of the equation of the trajectory must be inverted. The condition of stability is again (18).

(e) We study the motion in the electric potential

$$J = (\mu/\epsilon)\,[(\hbar^2/2\mu^2)(\alpha \tan^2(\theta/2] + \beta \tan(\theta/2) + \gamma) + V(r)]. \tag{25}$$

Equation (8) can be integrated with the new variable $\tan(\theta/2)$.

The trajectory is described by an equation of a rather unusual type:

$$i[\hbar^2(\gamma - \alpha + i\beta)/\mu^2 - b]^{-1/2} \arcsin \frac{[\hbar^2(\beta + 2i\alpha)/\mu^2]\tan(\theta/2) + [\hbar^2(2\gamma + i\beta)/\mu^2] - 2b}{[\tan(\theta/2) - i][(\hbar^4\beta^2/\mu^4) + 4(\hbar^2\alpha/\mu^2)(b - \hbar^2\gamma/\mu^2)]^{1/2}} + \text{c.c.} = -\,b^{-1/2} \arcsin \frac{b - Hr}{r(H^2 + ab)^{1/2}} + C^t. \tag{26}$$

Complex quantities are mixed to give a final real result; the problem (c) led to the same remark; there are no restrictions on the values of α, β, and γ.

(f) We study the motion in the electric potential

$$J = (\mu/\epsilon)\,[(\hbar^2/2\mu^2)(\alpha \cot^2(\theta/2) + \beta \cot(\theta/2) + \gamma) + V(r)]. \tag{27}$$

Since it follows from (25) by the substitution $\theta \to \pi - \theta$ Equation (8) indicates that the equation of the trajectory is obtained by carrying the same substitution in (26) after having inverted the sign of the first member. No restrictions about α, β, and γ.

(g) We study the motion in the electric potential

$$J = (\mu/\epsilon)[(\hbar^2/2\mu^2)(\alpha \tan^2\theta + \beta \tan\theta + \gamma) + V(r)] \tag{28}$$

The equation of the trajectory obviously follows from (23) by the substitution $\theta \to \pi/2 - \theta$, after inverting the sign of the first member. No restrictions about α, β, and γ.

2. *Elliptic integrations*

Let us make the equation of the trajectory (8) rational by a suitable change of variables. If the irrationality is of the third or of the fourth degree the trajectory may be written with the aid of elliptic functions. The potentials (1) for which $f(\theta)$ has the following values lead to elliptic integrals:

$$f(\theta) = (\alpha \cos^2\theta + \beta \cos\theta + \gamma)\cos^{-2}\theta \quad (\text{put } u = \cos\theta),$$

$$f(\theta) = (\alpha \sin^2\theta + \beta \sin\theta + \gamma)\sin^{-2}\theta \quad (\text{put } z = \sin\theta),$$

$$\left.\begin{aligned} f(\theta) &= (\alpha \cos\theta + \beta \sin\theta + \gamma)\cos^{-1}\theta \\ f(\theta) &= (\alpha \cos\theta + \beta \sin\theta + \gamma)\sin^{-1}\theta \\ f(\theta) &= (\alpha \cos\theta + \beta \sin\theta + \gamma \sin^2\theta + \delta \sin\theta\cos\theta + \epsilon) \end{aligned}\right\} \quad [\text{put } y = \tan(\theta/2)].$$

The list does not terminate here but we think it is of little interest to write it *in extenso*.

3. *Other possibilities of exact motion*

The conclusions of the preceding sections are valid with arbitrary initial conditions (see, however, Remarks 1 and 2). In this section we deal with exact motions allowed by suitable initial conditions. It was recently shown by Armenti and Havas[12] that an exact motion is sometimes possible outside the plane of symmetry $\theta = \pi/2$ when a monopole-prolate quadrupole potential acts on the particle. However, very special initial conditions are needed to this end. The authors noted that the conclusions are also valid when one considers noncentral potentials if in addition to an attractive radial force, there is a θ-component of the noncentral force directed away from the plane of symmetry. It must be pointed out that since the existence of such a movement depends on the initial conditions no equivalent can exist in the quantum formalism.

(a) We prove that the following exact motion is possible: $\dot\theta = 0, \dot\varphi = \omega = \text{const} \to \varphi = \omega t + \varphi_0$. The trajectory is thus a circle located in a plane at the distance $d = r\cos\theta$ from the plane of symmetry.

From the equations of motion[11]:

$$V'(r) - 2f(\theta)/r^3 = \omega^2 r \sin^2\theta,$$

$$f'(\theta)/r^3 = \omega^2 r \sin\theta\cos\theta.$$

We deduce

$$\omega^2 = f'(\theta)/(r^4 \sin\theta\cos\theta),$$

and

$$r^3 V'(r) = 2f(\theta) + f'(\theta)\tan\theta.$$

Such a motion is therefore possible provided $f'\cos\theta > 0$ and $r^3 V' - 2f > 0$. The first equation determines the angular velocity while the second connects r and θ, i.e., it gives the distance $d = r\cos\theta$ between the plane of the trajectory and the plane of symmetry. As an example, let us investigate the case $V(r) = Ar^{n-2}$. Simple algebraic calculations show that

$$d = \{[2f(\theta) + f'(\theta)\tan\theta]/A(n-2)\}^{1/n}\cos\theta,$$

$$\omega = \pm\,[f'(\theta)/\sin\theta\cos\theta]^{1/2}\{A(n-2)/[2f(\theta) + f'\tan\theta]\}^{2/n}.$$

(b) Another special motion is deduced from: $\dot\theta = 0$, $\dot\varphi = 0$. The trajectory is located on a straight line passing through the origin. Of course such a motion also requires special initial conditions. Furthermore, if we restrict ourselves to potentials of the type (1), we must ensure that $\mathbf{r}\wedge\gamma = 0$ which leads to $(-y, x, 0)f' = 0$. If $f' = 0$, $f = \text{const}$ is fulfilled the problem may be solved exactly like a one-dimensional problem on account of the fact that the potential remains central. An exact straight line motion in a noncentral potential is also possible along the Oz axis because in this case $x = y = 0$.

II. SCHRÖDINGER'S MECHANICS

We have already seen (see Remark 2) that the quantum problem involving potentials like (1) is soluble in three dimensions if $f(\theta)$ is given by (10), (11) or (12) and that it is soluble in two dimensions in all cases (10) to (16).

A. The three-dimensional problem

We use spherical coordinates r, θ, φ. Schrödinger's equation takes the form:

$$\frac{\partial^2\psi}{\partial r^2} + (2/r)\frac{\partial\psi}{\partial r} + (1/r^2)\frac{\partial^2\psi}{\partial\theta^2} + (\cot\theta/r^2)\frac{\partial\psi}{\partial\theta} - (m^2/r^2\sin^2\theta)\psi + \frac{2\mu}{\hbar^2} \times [E - \mu f(\theta)/r^2 - \mu V(r)]\psi = 0.$$

The variables can be separated in the usual way: $\psi = \exp(im\varphi)\,\Theta(\theta)R(r)$ (m is the usual magnetic quantum number; we assume $m > 0$; calculations are analogous when $m < 0$). One has

$$r^2R'' + 2rR' + (2\mu/\hbar^2)r^2(E - \mu V(r))R + sR = 0, \tag{29}$$

$$\Theta'' + \cot\theta\,\Theta' - (m^2/\sin^2\theta)\Theta - (2\mu^2/\hbar^2)f(\theta)\Theta - s\Theta = 0. \tag{30}$$

As in the Newtonian formalism, the radial motion does not depend on the term $f(\theta)/r^2$ present in the potential. We next investigate the two cases mentioned in Sec. I.A.

Case 1: $V = V_1 = -H/r$.

Equation (29) reduces to the radial equation of a hydrogen-like system. The energy levels are

$$E = -(\mu^3H^2/2\hbar^2)[n + 1/2 + (1/4 - s)^{1/2}]^{-2}, \tag{31}$$

where the parameter s may only take special values to be determined from (30).

Case 2: $V = V_2 = Kr^2$.

The radial equation (29) is the same as in the theory of the three-dimensional harmonic oscillator; the energy levels are given by

$$E = \hbar\sqrt{2K}\,[2n + 1 + (1/4 - s)^{1/2}], \tag{32}$$

where again s is quantized. Of course when $f(\theta) = 0$, $s = -l(l+1)$ and we obtain the classical formulas for the energy levels of the hydrogen atom or of the harmonic oscillator.

It only remains to solve the θ-equation (which does not depend on the choice $V = V_1$ or $V = V_2$) to discover the allowed s-values.

The θ-equation is exactly soluble by means of known transcendental functions only when $f(\theta)$ is given by (10), (11) or (12).

(a) The quantum motion in the electric potential (17).

The θ-equation (30) becomes

$$\Theta'' + \cot\theta\,\Theta' - (m^2/\sin^2\theta)\Theta - (\alpha\cos^2\theta + \beta\cos\theta + \gamma)\sin^{-2}\theta\,\Theta - s\Theta = 0. \tag{33}$$

We make the following substitutions:

$$v = \cos^2(\theta/2), \qquad \Theta = v^\rho(1-v)^\sigma T,$$

where

$$\rho = (1/2)(m^2 + \alpha - \beta + \gamma)^{1/2},$$

$$\sigma = (1/2)(m^2 + \alpha + \beta + \gamma)^{1/2},$$

$$v(1-v)T'' + [(2\rho + 1) - (2\rho + 2\sigma + 2)v]T' - [2\rho\sigma + s + \sigma + \beta/2 + 2\rho^2 + \rho - \alpha]T = 0.$$

We recognize the hypergeometric equation. The polynomial condition gives the allowed s-values. One finds ($k = 0, 1, 2, \cdots$)

$$T = F(-k, k + (m^2 + \alpha - \beta + \gamma)^{1/2} + (m^2 + \alpha + \beta + \gamma)^{1/2} + 1; 1 + (m^2 + \alpha - \beta + \gamma)^{1/2}; v),$$

$$\sqrt{1/4 - s} = -\alpha + (k + \rho + \sigma + 1/2)^2.$$

This relation must be introduced into (31) and (32) to obtain the energy levels when $V = V_1$ or $V = V_2$, respectively. In what follows we only consider $V = V_1$. (The other case is analogous.) The energy levels are by (31)

$$E = -(\mu^3H^2/2\hbar^2)\{n + 1/2 + \sqrt{-\alpha + [k + (1/2)(m^2 + \alpha - \beta + \gamma)^{1/2} + (1/2)(m^2 + \alpha + \beta + \gamma)^{1/2} + 1/2]^2}\}^{-2},$$

when $n,k=0,1,2,\cdots$.

(b) The quantum motion in the electric potential (20).

The θ-equation (30) becomes

$$\Theta''+\cot\theta\,\Theta'-(m^2/\sin^2\theta)\Theta-(\alpha\cos^4\theta+\beta\cos^2\theta+\gamma)\times\sin^{-2}\theta\cos^{-2}\theta\,\Theta-s\Theta=0. \quad (34)$$

We make the following substitutions:

$$w=\cos^2\theta,\quad \Theta=w^\rho(1-w)^\sigma T,$$

where

$$\rho=1/4+(1/4)(1+4\gamma)^{1/2},$$
$$\sigma=(1/2)(m^2+\alpha+\beta+\gamma)^{1/2},$$

$$w(1-w)T''+[(2\rho+1/2)-(2\rho+2\sigma+3/2)w]T'-(1/4)(s+8\rho\sigma+m^2+\beta+2\gamma+2\sigma+4\rho)T=0.$$

The solution is again the hypergeometric function

$$T=F(-k,k+1+(1/2)(1+4\gamma)^{1/2}+(m^2+\alpha+\beta+\gamma)^{1/2};1+(1/2)(1+4\gamma)^{1/2};w),$$
$$1/4-s=-\alpha+[2k+1+(1/2)(1+4\gamma)^{1/2}+(m^2+\alpha+\beta+\gamma)^{1/2}]^2,$$

with the energy levels:

$$E=-(\mu^3H^2/2\hbar^2)\{n+1/2+\sqrt{-\alpha+[2k+1+(1/2)(1+4\gamma)^{1/2}+(m^2+\alpha+\beta+\gamma)^{1/2}]^2}\}^{-2},\quad \text{where } n,k=0,1,2,\cdots.$$

(c) The quantum motion in the electric potential (22)

The θ-equation (30) becomes

$$\Theta''+\cot\theta\,\Theta'-(m^2/\sin^2\theta)\Theta-(\alpha\cot^2\theta+\beta\cot\theta+\gamma)\Theta-s\Theta=0.$$

We make the following substitutions:

$$z=e^{2i\theta},\quad \Theta=z^\sigma(1-z)^\tau T,$$

where

$$\sigma=(1/4)+(1/2)(1/4-\gamma-s+i\beta+\alpha)^{1/2},$$
$$\tau=(m^2+\alpha)^{1/2}.$$

The T-equation is again hypergeometric. One finds $(k=0,1,2,\cdots)$

$$s=(1/4)-\gamma+\alpha-\frac{(2k+1+2\sqrt{m^2+\alpha})^4-4\beta^2}{4(2k+1+2\sqrt{m^2+\alpha})^2},$$

$$T=F[-k,k+1+(1/4-\gamma-s+i\beta+\alpha)^{1/2}+2(m^2+\alpha)^{1/2};1+(1/4-\gamma-s+i\beta+\alpha)^{1/2};z].$$

The energy levels are given by (31) (we only consider the case $V=V_1$):

$$E=-(\mu^3H^2/2\hbar^2)\left[n+1/2+\sqrt{\gamma-\alpha+\frac{(2k+1+2\sqrt{m^2+\alpha})^4-4\beta^2}{4(2k+1+2\sqrt{m^2+\alpha})^2}}\right]^{-2},$$

where $n,k=0,1,2,\cdots$.

B. The two-dimensional problem

We use cylindrical coordinate r,θ,z. Schrödinger's equation is then written as

$$r^2\frac{\partial^2\psi}{\partial r^2}+r\frac{\partial\psi}{\partial r}+\frac{\partial^2\psi}{\partial\theta^2}+r^2\frac{\partial^2\psi}{\partial z^2}+(2\mu/\hbar^2)[Er^2-\mu r^2V(r)-\mu f(\theta)]\psi=0.$$

Variables can be separated in the usual way: $\psi=\exp(ip_zz/\hbar)R(r)\Theta(\theta)$. ($p_z$ is the z component of the momentum; it is a constant of the motion.) One has

$$r^2R''+rR'+(2\mu/\hbar^2)r^2[E-(p_z^2/2\mu)-\mu V(r)]R-s^2R=0, \quad (35)$$

$$\Theta''-(2\mu^2/\hbar^2)f(\theta)\Theta+s^2\Theta=0. \quad (36)$$

Once again the radial motion is independent of $f(\theta)$. $V=V_1$ or V_2 and the radial equation is analogous to that encountered in the problem of the two-dimensional hydrogen atom[12] or that of the two-dimensional oscillator. Energy levels are given by

Case 1: $V=V_1=-H/r$,
$$E=(p_z^2/2\mu)-2(\mu^3H^2/\hbar^2)(2n+2s+1)^{-2}, \quad (37)$$

Case 2: $V=V_2=Kr^2$,
$$E=(p_z^2/2\mu)+\hbar\sqrt{2K}(2n+s+1), \quad (38)$$

$n=0,1,2,\cdots$ while s may only take quantized values. These are found by solving (36).

The θ-equation is soluble in terms of known transcendental functions in seven cases: when $f(\theta)$ is given by (10)$\cdots$ or (16). Three of them have been treated in three dimensions. Therefore we shall omit the corresponding two-dimensional treatments.

(d) The quantum motion in the electric potential (24) (two-dimensional). The θ-equation (36) becomes

$$\Theta''-(\alpha\sin^2\theta+\beta\sin\theta+\gamma)\cos^{-2}\theta\,\Theta+s^2\Theta=0.$$

We make the following substitutions:

$$y=(1-\sin\theta)/2,\quad \Theta=y^\rho(1-y)^\sigma T,$$

where

$$\rho=1/4+1/4(1+4\alpha+4\beta+4\gamma)^{1/2},$$
$$\sigma=1/4+1/4(1+4\alpha-4\beta+4\gamma)^{1/2},$$

$$y(1-y)T''+[(2\rho+1/2)-(2\rho+2\sigma+1)y]T'-(1/2)(-2s^2+\rho+\sigma+4\rho\sigma+\gamma-\alpha)T=0.$$

The solution is hypergeometric:

$$T=F(-k,k+1+(1/2)(1+4\alpha+4\beta+4\gamma)^{1/2}+(1/2)(1+4\alpha-4\beta+4\gamma)^{1/2};1+(1/2)(1+4\alpha+4\beta+4\gamma)^{1/2};y),$$
$$s^2=-\alpha+[k+1/2+(1/4)(1+4\alpha+4\beta+4\gamma)^{1/2}+(1/4)(1+4\alpha-4\beta+4\gamma)^{1/2}]^2.$$

The energy levels follow from

$$E = (p_z^2/2\mu) - 2(\mu^3H^2/\hbar^2) \times \{2n + 1 + 2\sqrt{-\alpha + [k + 1/2 + (1/4)(1 + 4\alpha + 4\beta + 4\gamma)^{1/2} + (1/4)(1 + 4\alpha - 4\beta + 4\gamma)^{1/2}]^2}\}^{-2}$$

where $n, k = 0, 1, 2, \cdots$.

(e) The quantum motion in the electric potential (25) (two-dimensional). The θ-equation (36) becomes

$$\Theta'' - [\alpha \tan^2(\theta/2) + \beta \tan(\theta/2) + \gamma]\Theta + s^2\Theta = 0.$$

We make the following substitutions:

$$z = -e^{i\theta}, \quad \Theta = z^\rho(1-z)^\sigma T,$$

where

$$\rho = (s^2 + \alpha - i\beta - \gamma)^{1/2},$$

$$\sigma = 1/2 + (1/2)(1 + 16\alpha)^{1/2},$$

$$z(1-z)T'' + [(2\rho + 1) - (2\rho + 2\sigma + 1)z]T' - (2\rho\sigma + \sigma + 4\alpha - 2i\beta)T = 0.$$

The solution is hypergeometric:

$$T = F(-k, k + 2\rho + 1 + (1 + 16\alpha)^{1/2}; 2\rho + 1; z),$$

where

$$\rho^2 = -i\beta + (1/4)\{[k + (1/2) + (1/2)(1 + 16\alpha)^{1/2}]^4 - 4\beta^2\} \times [k + (1/2) + (1/2)(1 + 16\alpha)]^{-2}.$$

One finds

$$s^2 = -\alpha + \gamma + \frac{[k + (1/2) + (1/2)(1 + 16\alpha)^{1/2}]^4 - 4\beta^2}{4[k + (1/2) + (1/2)(1 + 16\alpha)^{1/2}]^2}.$$

The energy levels follow from

$$E = (p_z^2/2\mu) - 2(\mu^3H^2/\hbar^2)\left\{2n + 1 + 2\sqrt{\gamma - \alpha + \frac{[k + (1/2) + (1/2)(1 + 16\alpha)^{1/2}]^4 - 4\beta^2}{4[k + (1/2) + (1/2)(1 + 16\alpha)^{1/2}]^2}}\right\}^{-2}, \quad \text{where } n, k = 0, 1, 2, \cdots.$$

(f) The quantum motion in the electric potential (27) (two-dimensional). The θ-equation (36) becomes

$$\Theta'' - (\alpha \cot^2(\theta/2) + \beta \cot(\theta/2) + \gamma)\Theta + s^2\Theta = 0.$$

It deduces from the θ-equation of Sec. II. B e by the substitution $\theta \to \pi - \theta$. Therefore, the wavefunction is obtained through the same procedure while the energy levels are given by the same formula.

(g) The quantum motion in the electric potential (28) (two-dimensional). The θ-equation (36) becomes

$$\Theta'' - (\alpha \tan^2\theta + \beta \tan\theta + \gamma)\Theta + s^2\Theta = 0.$$

We make the following substitutions:

$$z = 1 + e^{2i\theta}, \quad \Theta = z^\rho(1-z)^\sigma T.$$

where

$$\rho = (1/2) + (1/2)(1 + 4\alpha)^{1/2},$$

$$\sigma = (1/2)(s^2 + \alpha - i\beta - \gamma)^{1/2},$$

$$z(1-z)T'' + [-(2\rho + 2\sigma + 1)z]T' - [2\rho\sigma + \rho + \alpha - (i\beta/2)]T = 0.$$

The solution is hypergeometric:

$$T = F(-k, k + 1 + (1 + 4\alpha)^{1/2} + (s^2 + \alpha - i\beta - \gamma)^{1/2}; 1 + (1 + 4\alpha)^{1/2}; z)$$

$$s^2 = \gamma - \alpha + \frac{[(1 + 4\alpha)^{1/2} + 1 + 2k]^4 - 4\beta^2}{4[(1 + 4\alpha)^{1/2} + 1 + 2k]^2},$$

with energy levels

$$E = (p_z^2/2\mu) - 2(\mu^3H^2/\hbar^2)\left\{2n + 1 + 2\sqrt{\gamma - \alpha + \frac{[(1 + 4\alpha)^{1/2} + 1 + 2k]^4 - 4\beta^2}{4[(1 + 4\alpha)^{1/2} + 1 + 2k]^2}}\right\}^{-2}, \quad \text{where } n, k = 0, 1, 2, \cdots.$$

C. Conditionally soluble quantum motions

In this section we shall investigate the solubility of Schrödinger's equation when more complicated potentials are considered, precisely those which lead to elliptic functions in the classical theory. Since these potentials are numerous we shall restrict ourselves to a special case in view of illustrating what we have called in a previous paper[10] the "conditional solubility".

We choose the special case

$$f(\theta) = (\hbar^2/2\mu^2)\,\alpha \cos^{-1}\theta$$

so that we investigate the quantum motion in the electric potential

$$J = (\mu/\epsilon)\,[\alpha(\hbar/2\mu^2)/(r^2 \cos\theta) + V(r)].$$

Considering the three-dimensional problem so that one has in spherical coordinates, the θ-equation (30) can be written as follows in spherical coordinates:

$$\Theta'' + \cot\theta\,\Theta' - m^2 \sin^{-2}\theta\,\Theta - \alpha \cos^{-1}\theta\,\Theta - s\Theta = 0.$$

We make the following substitutions:

$$u = \cos\Theta, \quad \Theta = (1-u^2)^{-m/2}\,T;$$

one finds

$$u(1-u)(-1-u)\,T'' + (2-2m)u^2\,T' + [(m^2-m+s)u+\alpha]\,T = 0.$$

This equation is of the general type

$$u(1-u)(\alpha-u)f'' + (au^2+bu+c)f' + (d+eu)f = 0.$$

We have studied it previously.[6] Polynomials solutions exist which ensure the integrability of $|\Theta|^2$ provided the four following conditions are fulfilled:[10]

$$\begin{cases} a+b+c = -j'(1-\alpha) \\ a\alpha^2+b\alpha+c = -j''\alpha(\alpha-1) \\ \qquad\qquad \text{(where } j' \text{ and } j'' \text{ are integers} \geqslant 0) \\ e = -n(n+a-1) \\ + \ a \text{ "continuant" condition (see Ref. 10).} \end{cases}$$

Then one has $T = z^{j'+1}(1-z)^{j''+1}P^{(\nu)}$, where $P^{(\nu)}$ denotes a polynominal of degree ν.

The first two conditions are satisfied if $j' = j'' = m-1$.

The third implies $m^2 - m + s = -(\nu+2m)(m+\nu+1)$ which gives the allowed s-values.

The continuant condition is expressed by the vanishing of a continuant of order $\nu+1$. We have previously seen[10] that each value of ν must be analyzed separately leading to a quantization of the parameter α entering into the definition of the potential. For example, when $\nu = 2$ the determinant is of order three:

$$\begin{vmatrix} d & c & 0 \\ e & d+b & 2(c-1) \\ 0 & a+e & d+2b \end{vmatrix} = 0.$$

In this case, the problem is soluble only if $\alpha = \pm 2\sqrt{4m+2}$. Performing the same operation for each value of ν we arrive at the list of the allowed values for α. Inversely α being fixed (among the allowed values of course) ν, m and s are also fixed so that arbitrary angular momentum states are automatically forbidden. We retrieve the conclusions of our preceding paper.[10]

If $\nu = 2$, one finds $s = -(m+2)(m+3)$ and the energy levels are

$$E = -(\mu^3H^2/2\hbar^2)[n+m+3]^{-2}.$$

Remark 3: This formula is analogous to the one giving the hydrogen spectrum except for the fact that the ground state and the first excited state are missing. Such a truncated hydrogen-like spectrum is found for every value of ν.

III. DISCUSSION AND CONCLUSION

In this paper we examined the various potentials of the type (1) which allow a complete integration of the equations of motion in both classical and quantum nonrelativistic mechanics. Our first conclusion is that the three-dimensional problem is completely soluble if the particle experiences the potentials (17), (20), or (22) while the two-dimensional problem is soluble when potentials (17), (20), (22), (24), (25), (27), or (28) are considered. To our knowledge these potentials were not treated before in the literature. It is interesting to compare the classical and the quantum treatments. As in our previous paper[10] it is possible to exhibit analogies from two different points of view:

(a) Firstly, there are some purely formal analogies: the resolution of Newton's equation and that of Schrödinger's equation offer many common points in spite of their well distinct origins. Variables separate in both equations for the same potentials. The changes of variables needed for the complete calculation are often identical. The analogy is sometimes very suggestive, e.g., the classical handling of potentials (22) or (25) leads to a trajectory whose equation contains complex quantities but so mixed that the overall result is real. The quantum equation leads to the same result: it is impossible to avoid the use of complex numbers in the calculation of the energy levels though the final expression is of course real. Beside the potentials mentioned above, there is a large class of potentials which allow a complete integration of the classical equation of motion in terms of elliptic functions. The corresponding problem in quantum mechanics leads to a conditional solubility analog to that previously encountered in a paper dealing with the motion in various magnetic fields.[10]

(b) Physical analogies also exist. Firstly we note that in classical mechanics potentials (17), (20), and (22) allow an exact solution whatever the choice of the axis of reference while potentials (24), (25), (27), (28) are soluble only for a special choice of these axes. In quantum mechanics the potentials of the first category are soluble in three dimensions (also in two) while those of the second category are soluble in two dimensions only. A Schrödinger's equation soluble by means of elementary transcendental functions (with energy levels) corresponds in the classical theory to a bounded trajectory expressible by means of circular functions: the parameters entering into the definition of the potential are submitted to analogous conditions in both mechanics in order to warrant a stable motion. Before terminating let us illustrate this last point with an example. Consider the motion in the potential (17); the quantum solution is given in Sec. II. A(a). The formula which determines the energy levels is only valid under the conditions

$$\begin{aligned} &\alpha - \beta + \gamma \geqslant 0, \\ &\alpha + \beta + \gamma \geqslant 0. \end{aligned} \tag{39}$$

It is not difficult to show that these conditions are sufficient to ensure the stability of the classical trajectory. Let us return to Sec. I. B(a). It is easily seen that the θ-integration leads to circular functions only if $b + \alpha\hbar^2/\mu^2 > 0$. Otherwise, the quadrature leads to logarithms and the trajectory spirals to $r = 0$ or to infinity. Since $b - 2f(\theta) = P^2 > 0$ it is sufficient that

$$b + \alpha\hbar^2/\mu^2 \geqslant b - 2f(\theta), \quad \text{where } f(\theta) \text{ is given by (10).}$$

After some reductions one obtains the condition

$$\alpha + \beta \cos\theta + \gamma \geqslant 0$$

to be compared with (39).

In conclusion, the connection between the classical and the quantum treatments of the problem here investigated can be stated as follows: when the classical trajectory is stable ($0 < r_{\min} < r < r_{\max}$) and when it is expressible by means of circular functions, the corresponding quantum problem is exactly soluble in terms of known functions and the energy spectrum contains a discrete part. If the trajectory is stable but can only be expressed by means of elliptic functions, the connection with the classical motion disappears. However, in that case the example treated in Sec. II. C indicates that the quantum energy spectrum may be discrete if the parameters entering in the definition of the potential are limited to discrete values. Finally, it must be pointed out that exact motions in classical mechanics sometimes occur when very special initial conditions are imposed. In this case no equivalence seems to exist in quantum mechanics.

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