GENERALITÀ

The two-body relativistic interaction in recursive dynamics.

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SUMMARY. – *Recursive dynamics allows an accurate numerical resolution of the relativistic two-body problem. That is in contrast with the classical theory, which is only able to write an inextricable system of difference-differential equations.*

1. Introduction

Four fundamental interactions are known in physics: gravitational, electromagnetic, strong and weak. In each case, the two-body interaction is expected to play the central role. Surprisingly enough, that fundamental problem has never been solved accurately neither theoretically nor numerically. Of course, an exact solution of the gravitational problem has been found in the Newtonian limit but no equivalent solution is known in the frame of the special theory of relativity (1). Only very special cases have been investigated: the classical case of an infinitely heavy centre and the case of concentric circular trajectories (2). In the general case, the calculations are so intricate that there is little hope to solve them one day even numerically. Solutions to the equations of such a fundamental problem cannot be obtained indicating that something is wrong in the traditional approach of the fundamental interaction problem.

The two-body problem is classically solved without difficulty in the Newtonian limit, which is based on the concept of instantaneous force. This approximation is acceptable on the scale of our solar system, however the weakness of the method is that interaction cannot propagate with an infinite speed. The replacement of the instaneous force by a time-delayed force is known to lead to an unstable model (3) so that there is no hope to save the Newtonian picture.

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What we said about the gravitational interaction is also true with the electromagnetic interaction. In fact, the situation is even worse in that case. Because the Maxwell equations are naturally relativistically invariant, the theory of special relativity might be considered as their natural frame. Unfortunately no satisfactory solution can be found because of two major objections:

– If Maxwell's theory of electromagnetism accurately describes the behaviour of the electromagnetic field in vacuum, difficulties, first pointed out by Feynmann and Wheeler, arise when one tries to calculate the trajectories of charged particles in these fields. One finds, for example, that an electron moving in the field of another charged particle would experience a self-acceleration as the consequence of the retarded interaction. A disturbing consequence is the well-known instability of the relativistic two-body system. Feynmann and Wheeler have tried to remove the difficulty by modifying the Maxwell-Lorentz equations. However their idea which consists in mixing, in equal proportions, advanced and retarded interactions seems almost an *ad hoc* trick.

– Even if one adopts the ideas of these authors, the problem remain completely unsolvable, even numerically, because of the high complexity of the difference-differential equations which describe the system.

The current models being highly unsatisfactory, this paper reports an improved model, which leads to a better understanding of the mechanism of the two-body interaction.

The origin of the above mentioned troubles is believed to be found in the bad idea of having generalised in electromagnetism the concept of force which is inheritited from Newton's mechanics. It is our purpose to show that a completely different approach might avoid all those difficulties. We first recall the fundamental principles of a new methodology already presented in two previous papers (4, 5) dealing with the special case of the one-dimensional motion. Then we shall write and analyse the equations of motion in the general case.

2. The general picture of the two-body interaction in recursive dynamics.

Two interacting particles, of respective rest masses, m and m', are located at the positions, $\vec{r}_0$ and $\vec{r}'_0$, at the initial time, $t_0=t'_0$. They initially present positive (resp. negative) mass deviations, δ_0 and δ'_0, in the case of a repulsive (resp. attractive) interaction, so that their initial dynamical masses are respectively equal to $m+\delta_0$ and $m'+\delta'_0$.

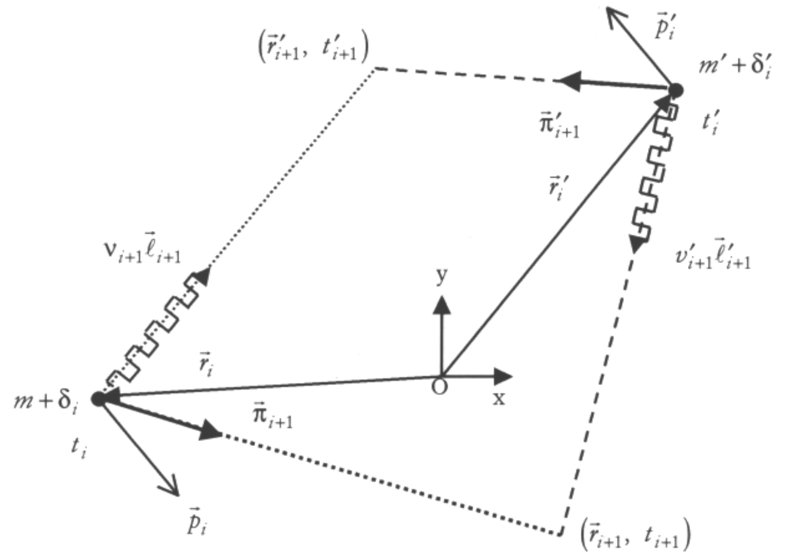

FIG. 1

When the particles are released, with initial momenta $\vec{p}_0$ and $\vec{p}'_0$, they each immediately emit a positive (resp. negative) energy photon in the direction $\vec{\ell}_1$ and $\vec{\ell}'_1$ of the future position of the other particle, of such positive (resp. negative) frequencies, ν_1 and ν'_1, that they instantaneously recover their original rest mass. The momentum of each of the particles suddenly changes in agreement with the momentum conservation law and their new values are denoted $\vec{\pi}_1$ and $\vec{\pi}'_1$. Each particle absorbs the photon emitted by his partner and then reemits a new photon in the direction of its future position. A repulsive (resp. attractive) motion results from the laws of conservation of energy and momentum. Each particle experiences successive cycles, emission and absoption, without interruption. Figure 1 shows the i[th] cycle.

3. The equations of motion.

The complete equations of motion are easily written as:

[1a,b]
$$\begin{cases} w_i = \sqrt{(m+\delta_i)^2 + p_i^2} = \sqrt{m^2 + \pi_{i+1}^2} + \nu_{i+1} \\ \vec{p}_i = \vec{\pi}_{i+1} + \nu_{i+1}\vec{\ell}_{i+1} \end{cases}$$

[2a,b]
$$\begin{cases} w'_i = \sqrt{(m'+\delta'_i)^2 + p_i'^2} = \sqrt{m'^2 + \pi_{i+1}'^2} + \nu'_{i+1} \\ \vec{p}'_i = \vec{\pi}'_{i+1} + \nu'_{i+1}\vec{\ell}'_{i+1} \end{cases}$$

$$[3a,b] \qquad \begin{cases} \vec{r}_{i+1} = \vec{r}_i + \vec{\pi}_{i+1}(t_{i+1} - t_i) / \sqrt{m^2 + \pi_{i+1}^2} \\ \vec{r}'_{i+1} = \vec{r}_i + \vec{\ell}_{i+1}(t'_{i+1} - t_i) \end{cases}$$

$$[4a,b] \qquad \begin{cases} \vec{r}'_{i+1} = \vec{r}'_i + \vec{\pi}'_{i+1}(t'_{i+1} - t'_i) / \sqrt{m'^2 + \pi'^2_{i+1}} \\ \vec{r}_{i+1} = \vec{r}'_i + \vec{\ell}'_{i+1}(t_{i+1} - t'_i) \end{cases}$$

$$[5a,b] \qquad \begin{cases} \sqrt{m^2 + \pi_{i+1}^2} + \nu'_{i+1} = \sqrt{(m + \delta_{i+1})^2 + p_{i+1}^2} = w_{i+1} \\ \vec{p}_{i+1} = \vec{\pi}_{i+1} + \nu'_{i+1}\vec{\ell}'_{i+1} \end{cases}$$

$$[6a,b] \qquad \begin{cases} \sqrt{m'^2 + \pi'^2_{i+1}} + \nu_{i+1} = \sqrt{(m' + \delta'_{i+1})^2 + p'^2_{i+1}} = w'_{i+1} \\ \vec{p}'_{i+1} = \vec{\pi}'_{i+1} + \nu_{i+1}\vec{\ell}_{i+1} \end{cases}$$

The motion is confined in the plane defined by the initial position vectors, $\vec{r}_0$ and $\vec{r}'_0$. Each vector has only two non-vanishing components and the vectors, $\vec{\ell}_{i+1}$ and $\vec{\ell}'_{i+1}$, are unitary. Equations [1ab] to [6ab] form a recursive system equivalent to 20 scalar equations. The unknowns are the (i+1)-indexed variables and they may be divided in two categories: 12 state variables, ($\vec{r}_{x,i+1}$, $\vec{r}_{y,i+1}$, $\vec{r}'_{x,i+1}$, $\vec{r}'_{y,i+1}$, $\vec{p}_{x,i+1}$, $\vec{p}_{y,i+1}$, $\vec{p}'_{x,i+1}$, $\vec{p}'_{y,i+1}$, t_{i+1}, t'_{i+1}, δ_{i+1}, δ'_{i+1}), associated to the particles and 8 intermediate variables ($\vec{\pi}_{x,i+1}$, $\vec{\pi}_{y,i+1}$, $\vec{\pi}'_{x,i+1}$, $\vec{\pi}'_{y,i+1}$, $\vec{\ell}_{i+1}$, $\vec{\ell}'_{i+1}$, ν_{i+1}, ν'_{i+1}) associated to the mediating photons. Solving such an algebraic system might appear a formidable task. However it is easily shown that a numerical solution is possible by programming the complete system in the *FindRoot* instruction of Wolfram's Mathematica language (6). The resolution requires the knowledge of twelve initial conditions corresponding to the state variables set. This poses no problem except for the mass deviations δ_0 and δ'_0. Their global intensity, $\delta_0+\delta'_0$, clearly depends of the strength of the coupling between the interacting particles but it is not clear how one should share it between the two particles.

This difficult problem will be refered as «the sharing law problem» and remains open because of a lack of information which is probably related to the generalisation of the third Newton's law. The fact that action and reaction are no more opposite in the general case does not simplify the task. In this paper, we prove the relativistic invariance of the fundamental equations and we solve the problem of the sharing law in the Newtonian limit. It is hoped that the general sharing law will be soon discovered leading to the complete – at least numerical – solution of the general two-body interaction in both gravitational and electromagnetic cases.

4. The relativistic invariance of the equations of motion.

The relativistic invariance of the equations of motion is most easily verified after their transcription in the formalism of quadrivectors. In the following, we label eight 4-vectors by a capital letter, namely:

$$\Pi = (\pi_x, \pi_y, \pi_z, \sqrt{m^2 + \pi^2})$$
$$\Pi' = (\pi'_x, \pi'_y, \pi'_z, \sqrt{m'^2 + \pi'^2})$$
$$N = (\nu\ell_x, \nu\ell_y, \nu\ell_z, \nu)$$
$$N' = (\nu'\ell'_x, \nu'\ell'_y, \nu'\ell'_z, \nu')$$
$$R = (x, y, z, t)$$
$$R' = (x', y', z', t')$$
$$P = (p_x, p_y, p_z, \sqrt{(m+\delta)^2 + p^2})$$
$$P' = (p'_x, p'_y, p'_z, \sqrt{(m'+\delta')^2 + p'^2})$$

With these notations, the system may be rewritten in a condensed form. Separating the kinematical and the dynamical equations leads to:

Dynamical equations:

$$P_i = \Pi_{i+1} + N_{i+1}$$
$$P'_i = \Pi'_{i+1} + N'_{i+1}$$
$$P_{i+1} = \Pi_{i+1} + N'_{i+1}$$
$$P'_{i+1} = \Pi'_{i+1} + N'_{i+1}$$

Kinematical equations:

$$R'_{i+1} = R_i + N_{i+1}\frac{t'_{i+1} - t_i}{\nu_{i+1}}$$
$$R_{i+1} = R'_i + N'_{i+1}\frac{t_{i+1} - t'_i}{\nu'_{i+1}}$$
$$R_{i+1} = R_i + \Pi_{i+1}\frac{t_{i+1} - t_i}{\sqrt{m^2 + \pi^2_{i+1}}}$$
$$R'_{i+1} = R'_i + \Pi'_{i+1}\frac{t'_{i+1} - t'_i}{\sqrt{m'^2 + \pi'^2_{i+1}}}$$

Let us now consider a simple Lorentz transform with the relative velocity, $\vec{u}$, of the observer parallel to the x-axis. If the notation q^* is adopted for a variable q when it is viewed by the moving observer, it is well known that the transformation law of quadrivectors is written as:

$$V^*=TV$$

with

$$T=\begin{bmatrix}\beta & 0 & 0 & -\beta u\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ -\beta u & 0 & 0 & \beta\end{bmatrix}$$

where

$$\beta=1/\sqrt{1-u^2}$$

The dynamical equations are obviously invariant and consequently classical formulas of the Döppler effect or of the aberration effect can be observed:

$$\nu^*=\beta\nu\left(1-u\ell_x\right)$$

$$\ell_x^*=\frac{\ell_x-u}{1-u\ell_x}\qquad \ell_y^*=\frac{\ell_y}{\beta\left(1-u\ell_x\right)}\qquad \ell_x^*=\frac{\ell_z}{\beta\left(1-u\ell_x\right)}$$

Similar formulas are retrieved for the π-momenta of the particles:

$$\pi_x^*=\beta(\pi_x-u\sqrt{m^2+\pi^2})\qquad \pi_y^*=\pi_y\qquad \pi_z^*=\pi$$

$$\sqrt{m^2+\pi^{*2}}=\beta(\sqrt{m^2+\pi^2}-u\pi_x)$$

The invariance of the kinematical equations is less immediate. It is the consequence of the following relations, which are easily derived from the Lorentz transformation:

$$\frac{t_{i+1}'^*-t_i^*}{\nu_{i+1}^*}=\frac{t_{i+1}'-t_i}{\nu_{i+1}}$$

and

$$\frac{t_{i+1}^*-t_i^*}{\sqrt{m^2+\pi_{i+1}^{*2}}}=\frac{t_{i+1}-t_i}{\sqrt{m^2+\pi_{i+1}^2}}$$

The crucial point is that the transformation laws for the p-momenta are written as

$$\begin{cases}p_x^*=\beta\left(p_x-u\sqrt{(m+\delta)^2+p^2}\right)\\ \sqrt{\left(m+\delta^*\right)^2+p^{*2}}=\beta\left(\sqrt{(m+\delta)^2+p^2}-up_x\right)\end{cases}$$

A little algebra then leads to the conclusion that the mass deviations are relativistically invariant, i.e., $\delta^*=\delta$. In other words, the sharing law must be relativistically invariant as expected for a true law. Its general form is however not known because of the complexity of the fundamental system. It is known that a dynamical problem is solved when one succeeds to find all its dynamical invariants. A system like equations 1 to 6 possesses twelve degrees of freedom corresponding to the twelve initial conditions, which are necessary for its solution. Each initial condition may be considered as a potential non-autonomous (i.e. time dependent) invariant. Eliminating the time between all of them leads to, at least in principle, eleven autonomous invariants. Invariants such as energy and angular momentum are immediate but the remaining invariants cannot be found without an in-depth analysis of the whole system. The problem is so difficult than it shall remain partially open.

5. Elementary constants of the motion.

The simple inspection of the equations of evolution reveals, more or less evidently, the existence of four constants of the motion, which are the total energy, the total momentum, the total angular momentum and the velocity of the energy centre of the system, which will be discussed here below.

a) The total momentum of the system is conserved according one of the equivalent expressions:

[7] $$\vec{p}_i + \vec{p}'_i = \vec{\pi}_{i+1} + \vec{\pi}'_{i+1} + \nu_{i+1}\vec{\ell}_{i+1} + \nu'_{i+1}\vec{\ell}'_{i+1} = \vec{P}$$

A natural consequence is that the equations are simpler if one works in the zero-momentum frame (ZMF), in which P=0.

b) The total energy is also obviously conserved in agreement with:

[8] $$\begin{aligned} w_i + w'_i &= \sqrt{(m+\delta_i)^2 + p_i^2} + \sqrt{(m'+\delta'_i)^2 + p_i'^2} \\ &= \nu_{i+1} + \nu_{i+1} + \sqrt{m^2 + \pi_{i+1}^2} + \sqrt{m'^2 + \pi_{i+1}'^2} = W \end{aligned}$$

c) The total angular momentum, $\vec{J}$, is also conserved and can be written as:

[9] $$\vec{r}_i \times \vec{p}_i + \vec{r}_i' \times \vec{p}'_i = \vec{J}$$

Equ. [9] is less obvious and can be demonstrated by calculating successively Eq. 10:

[10] $$\begin{aligned} &\vec{r}_i \times \vec{\mathrm{p}}_\mathrm{i} + \vec{r}_i' \times \vec{p}'_i = \nu_{i+1}\vec{r}_i \times \vec{\ell}_{i+1} + \vec{r}_i \times \vec{\pi}_{i+1} + \nu'_{i+1}\vec{r}_i' \times \vec{\ell}'_{i+1} + \vec{r}_i' \times \vec{\pi}'_{i+1} \\ &\vec{r}_{i+1} \times \vec{p}_{i+1} + \vec{r}'_{i+1} \times \vec{\mathrm{p}}'_{i+1} = \nu'_{i+1}\vec{r}_{i+1} \times \vec{\ell}'_{i+1} + \vec{\mathrm{r}}_{i+1} \times \vec{\pi}_{i+1} + \\ &+\nu_{i+1}\vec{r}'_{i+1} \times \vec{\ell}_{i+1} + \vec{r}'_{i+1} \times \vec{\pi}'_{i+1} \end{aligned}$$

The difference of these equations leads to a vanishing result because of equations [3] and [4].

d) Another constant of the motion is associated to the uniform rectilinear motion of the centre of energy (CE) of the system. A correct definition of the

CE is possible in this model because the positions of the particles and the mediating bosons are known at all time. The following quantity is invariant:

$$[11]\quad \begin{aligned} & w_i\vec{r}_i + w'_i\vec{r}'_i - t_i\vec{p}_i - t'_i\vec{p}'_i = \\ & = \sqrt{(m+\delta_i)^2 + p_i^2}\,\vec{r}_i + \sqrt{(m'+\delta'_i)^2 + p_i'^2}\,\vec{r}'_i - t_i\vec{p}_i - t'_i\vec{p}'_i = \vec{\sigma} \end{aligned}$$

That result can be proved by establishing that the following recursion holds:

$$w_i\vec{r}_i + w'_i\vec{r}'_i - t_i\vec{p}_i - t'_i\vec{p}'_i = w_{i+1}\vec{r}_{i+1} + w'_{i+1}\vec{r}'_{i+1} - t_{i+1}\vec{p}_{i+1} - t'_{i+1}\vec{p}'_{i+1}$$

To do so each $(i+1)$-indexed variable must be replaced by its value in Eqs. [3] to [6]. If the origin of the reference axes coincides with the CE, which is possible with no loss of generality, than the constant vector, $\vec{\sigma}$, is equal to zero.

Five additional autonomous constants of the motion must exist. Some of them are expected to generalise the well-known Runge-Lenz vector of the classical Newtonian theory. Their discovery would lead to the complete analytical solution of the two-body interaction problem. The task is so difficult that it was left unsolved at this early stage of the study. We now concentrate on the numerical solution, which is fortunately accessible.

6. Numerical solution of the two-body problem.

Two methods are available: one is semi-analytical and the other is purely numerical.

6.1 – *Semi-analytical method.*

Eliminating $\vec{\pi}_{i+1}$ (resp. $\vec{\pi}'_{i+1}$) between Eqs. [1a-b] (resp. [2a-b]), one finds:

$$[12a,b]\quad \nu_{i+1} = \frac{z_i}{2w_i - 2\vec{p}_i \cdot \vec{\ell}_{i+1}} \quad \text{and} \quad \nu'_{i+1} = \frac{z'_i}{2w'_i - 2\vec{p}'_i \cdot \vec{\ell}'_{i+1}}$$

where, for the sake of brevity, z_i and z'_i are defined as:

$$z_i = 2m\delta_i + \delta_i^2 \quad \text{and} \quad z'_i = 2m'\delta'_i + \delta_i'^2$$

The following relations are then deduced:

$$[13a,b]\quad \begin{aligned} \vec{\pi}_{i+1} &= \frac{(2w_i - 2\vec{p}_i \cdot \vec{\ell}_{i+1})\vec{p}_i - z_i\vec{\ell}_{i+1}}{2w_i - 2\vec{p}_i \cdot \vec{\ell}_{i+1}} \\ \vec{\pi}'_{i+1} &= \frac{(2w'_i - 2\vec{p}'_i \cdot \vec{\ell}'_{i+1})\vec{p}'_i - z'_i\vec{\ell}'_{i+1}}{2w'_i - 2\vec{p}'_i \cdot \vec{\ell}'_{i+1}} \end{aligned}$$

[14a,b]
$$\sqrt{m^2+\pi_{i+1}^2}=\frac{2w_i^2-z_i-2w_i\vec{p}_i\cdot\vec{\ell}_{i+1}}{2w_i-2\vec{p}_i\cdot\vec{\ell}_{i+1}}$$
$$\sqrt{m'^2+\pi_{i+1}'^2}=\frac{2w_i'^2-z_i'-2w_i'\vec{p}_i'\cdot\vec{\ell}_{i+1}'}{2w_i'-2\vec{p}_i'\cdot\vec{\ell}_{i+1}'}$$

and finally:

[15a,b]
$$\frac{\vec{\pi}_{i+1}}{\sqrt{m^2+\pi_{i+1}^2}}=\frac{(2w_i-2\vec{p}_i\cdot\vec{\ell}_{i+1})\vec{p}_i-z_i\vec{\ell}_{i+1}}{2w_i^2-z_i-2w_i\vec{p}_i\cdot\vec{\ell}_{i+1}}$$
$$\frac{\vec{\pi}_{i+1}'}{\sqrt{m'^2+\pi_{i+1}'^2}}=\frac{(2w_i'-2\vec{p}_i'\cdot\vec{\ell}_{i+1}')\vec{p}_i'-z_i'\vec{\ell}_{i+1}'}{2w_i'^2-z_i'-2w_i'\vec{p}_i'\cdot\vec{\ell}_{i+1}'}$$

In the following step, $\vec{r}_{i+1}$ and $\vec{r}'_{i+1}$ are eliminated between Eqs. [3] and [4]. Then, a simple vector product eliminates the times, t_{i+1} and t'_{i+1}. It only remains two scalar equations for the director coefficients of the mediating bosons:

[16]
$$\left(\vec{r}_i-\vec{r}_i'\right)\times\left[\left(2w_i-2\vec{p}_i\cdot\vec{\ell}_{i+1}\right)\vec{p}_i-z_i\vec{\ell}_{i+1}-\left(2w_i^2-z_i-2w_i\vec{p}_i\cdot\vec{\ell}_{i+1}\right)\vec{\ell}_{i+1}'\right]=\left(t_i-t_i'\right)\vec{\ell}_{i+1}'\times\left[\left(2w_i-2\vec{p}_i\cdot\vec{\ell}_{i+1}\right)\vec{p}_i-z_i\vec{\ell}_{i+1}\right]$$

[17]
$$\left(\vec{r}_i-\vec{r}_i'\right)\times\left[\left(2w_i'-2\vec{p}_i'\cdot\vec{\ell}_{i+1}'\right)\vec{p}_i'-z_i'\vec{\ell}_{i+1}'-\left(2w_i'^2-z_i'-2w_i'\vec{p}_i'\cdot\vec{\ell}_{i+1}'\right)\vec{\ell}_{i+1}\right]=\left(t_i-t_i'\right)\vec{\ell}_{i+1}\times\left[\left(2w_i'-2\vec{p}_i'\cdot\vec{\ell}_{i+1}'\right)\vec{p}_i'-z_i'\vec{\ell}_{i+1}'\right]$$

Since the vectors, $\vec{\ell}_{i+1}=(\vec{\ell}_{x,i+1},\ \vec{\ell}_{y,i+1})$ and $\vec{\ell}'_{i+1}=(\vec{\ell}'_{x,i+1},\ \vec{\ell}'_{y,i+1})$ are unitary, one has two supplementary conditions:

[18a,b]
$$\vec{\ell}_{i+1}\cdot\vec{\ell}_{i+1}=1 \quad\text{and}\quad \vec{\ell}_{i+1}'\cdot\vec{\ell}_{i+1}'=1$$

Equations [16] to [18] form a coupled system for the unknowns, $\vec{\ell}_{x,i+1}$, $\vec{\ell}_{y,i+1}$, $\vec{\ell}'_{x,i+1}$, $\vec{\ell}'_{y,i+1}$. A classical decoupling procedure leads to four polynomial equations of degree eight. It can be shown that only one root is convenient for a complete verification of the fundamental Eqs. [1] to [6]. That procedure is however not the simplest available and it may be shown that a completely numerical approach is quicker.

6.2 – *Purely numerical method.*

The fundamental system of Eqs. [1] to [6] may be programmed in Wolfram's Mathematica language with the aid of a simple *FindRoot* instruction. The solution is quickly computed recursively in a stable way. The final result may be displayed in any desired way. Here is a numerical example which considers two interacting particles of respective rest masses m=1 and m'=3. There initial positions, momenta and mass deviations are as follows:

m=1; m'=3; x_0=–1; y_0=0; x'_0=–$x_0 w_0/w'_0$; y'_0=–$y_0 w_0/w'_0$; t_0=0; t'_0=0;
$p_{x,0}$=–$p'_{x,0}$=0; $p_{y,0}$=–$p'_{x,0}$=*alpha*; z_0=–2*alpha*^{2}; z'_0=$m' z_0/m$;
(**Comment: two numerical values of alpha are chosen below**)

The initial mass deviations have been shared according to the approximate law,

$$z'_0 = \frac{m'}{m} z_0.$$

Modifying that condition would not alter the trajectories very much. Mathematica is then able to compute the successive positions of both particles. The corresponding trajectories, in the (x, y) plane are displayed in two cases, alpha = 1/1000 and alpha = 1/7.

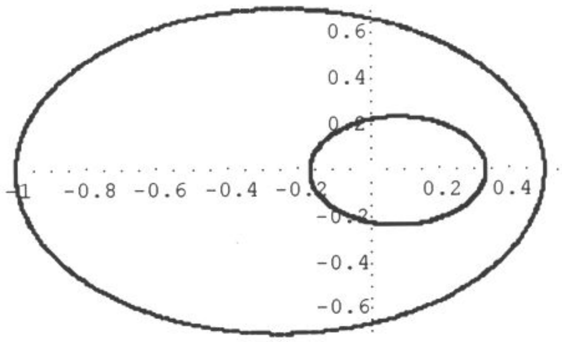

FIG. 2
alpha=1/1000

Figure 2 corresponds to a non-relativistic system since the ratio v/c is lower than 10^{-3}. One retrieves the Newtonian periodic elliptic trajectories.

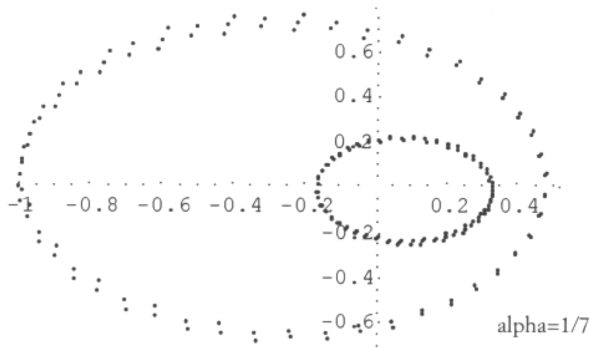

FIG. 3
alpha = 1/7

Figure 3 corresponds to a relativistic system since the ratio v/c is of the order 10^{-1}. Precessing trajectories are found, as expected.

7. The Newtonian limit.

If one agrees that the Newtonian trajectories are valid in the non-relativistic limit, we prove that the approximate sharing law may be written as:

[19] $$z'_0 / z_0 \approx m' / m$$

Eliminating the p-momenta between equations [I-1a] and [I-1b] and similarly between equations [2], [5] and [6] leads to four equations:

[20] $$z_i = 2\nu_{i+1}\left(\sqrt{m^2 + \pi_{i+1}^2} - \vec{\pi}_{i+1} \cdot \vec{\ell}_{i+1}\right)$$

[21] $$z'_i = 2\nu'_{i+1}\left(\sqrt{m'^2 + \pi'^2_{i+1}} - \vec{\pi}'_{i+1} \cdot \vec{\ell}'_{i+1}\right)$$

[22] $$z_{i+1} = 2\nu'_{i+1}\left(\sqrt{m^2 + \pi_{i+1}^2} - \vec{\pi}_{i+1} \cdot \vec{\ell}'_{i+1}\right)$$

[23] $$z'_{i+1} = 2\nu_{i+1}\left(\sqrt{m'^2 + \pi'^2_{i+1}} - \vec{\pi}'_{i+1} \cdot \vec{\ell}_{i+1}\right)$$

Two additional useful relations are:

[24] $$\nu_{i+1} + \nu'_{i+1} = W - \sqrt{m^2 + \pi_{i+1}^2} - \sqrt{m'^2 + \pi'^2_{i+1}}$$

and:

[25] $$2W(\nu'_{i+1} - \nu_{i+1}) = z_{i+1} - z_i - z'_{i+1} + z'_i$$

Eq. [24] is simply the conservation of energy and Eq. [25] is the consequence of an obvious mixing of Eqs. [20] to [24] with [1b] and [2b] taken into account.

In the low velocity limit it is immediately apparent that the radicals are the dominant terms in Eqs. [20] to [24] and that they are of the order of magnitude of m and m' respectively. One may therefore write:

[26] $$m'^2 z_i z_{i+1} \approx m^2 z'_i z'_{i+1}$$

The same recurrence exactly holds in the one-dimensional case as shown in our earlier paper (5). Its solution is easily written in terms of the initial conditions:

$$\begin{cases} z_{2i} / z'_{2i} \approx z_0 / z'_0 \\ z_{2i+1} / z'_{2i+1} \approx \left(m^2 z'_0 / m'^2 z_0\right) \end{cases}$$

At this stage, the problem of the balance between z_0 and z'_0 remains entirely open. We have verified numerically that one retrieves, in the classical limit, the results of the Newton theory if, and only if, one opts for what we shall call the classical sharing law:

$$z'_0 \approx \frac{m'}{m} z_0$$

This seems to indicate that the simultaneity criterion, $t'_{2i}=t_{2i}$, is not valid as initially thought in the context of the one-dimensional problem. Both in the classical limit and in the one-dimensional case, the correct criterion seems to be:

[27] $$z'_i = \frac{m'}{m} z_i \qquad \forall i$$

In the general case of higher velocities numerical evidences indicate that criterion [27] fails. It becomes velocity dependent in some relativistically invariant way that has not yet been discovered and that leaves the problem partially open. To solve it, a generalised form of Eq. [27] is needed, which probably implies the discovery of the complete invariants of the system.

7. Conclusion

There is a serious perspective to solve the general relativistic two-body problem in the frame of the recursive dynamics. Though the evolution equations of the system are rather lengthy, about twenty scalar equations, no difficulty has been encountered to solve them, at least numerically. The only remaining problem is a missing conjecture, which would allow sharing adequately the global mass defect of the system between the particles. It has been found that the law,

$$z_i' = \frac{m'}{m} z_i ,$$

is consistent with Newton's third law in the classical limit, and this law is called the classical sharing law. An in-depth theoretical study of the fundamental evolution equations, including a complete discussion of the dynamical invariants of the system, will be necessary to extend the classical sharing law to the general case.

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