Unified Interaction Through Boson Exchange: an essay on recursive dynamics

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SUMMARY. - *Boson exchange theory is directly applied to the two-body interaction. We have verified the long range 1/r behaviour of Coulomb's potential law when the rest mass of the mediating boson is zero. However, a short range discrepancy has been obtained and is discussed. We have also found an unexpected interaction law when the rest mass of the mediating boson is finite. An application, which concerns Keplerian motion, is given. This application seems to indicate that recursive dynamics is able to solve various problems of classical mechanics of the mass point by purely algebraic means. The extension of the model to the other types of interactions seems possible so that a door is open in the direction of the unified treatment of the interaction in physics.*

I. Introduction

Although this paper might be considered unusual, it leads to a pleasant pictorial way of describing the two body interaction. Moreover, it allows one to reconsider our approach of the fundamental laws of physics. Thirdly it illustrates the power of Mathematica (1) to solve difficult algebraic problems.

Interactions, in classical physics, are usually described in terms of forces. However, neither atomic nor molecular physics nor cosmology make use of them. This seems to indicate that the concept of force, though useful in every day life, is not essential nor indispensable.

For electrodynamics, one generally credits Feynmann with the idea that any interaction could be seen as the result of the exchange of virtual bosons (2, 3), i.e., in case I, zero rest mass bosons, such as photons or gravitons, for long-range electrostatic and gravitational interactions and, in case II, finite rest mass bosons exchange for nuclear short-range interactions. In short, interacting particles are supposed to repeatedly exchange virtual bosons

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so that they modify their own energy and momentum at each emission or absorption, in accord with the classical conservation laws of physics.

As is often the case in quantum mechanics, i.e., the well known example of the spin magnetic moment of a «rotating charged object», such a naive picture may not be taken too literally. The adjective «virtual» reminds us that it is not necessary, nor necessarily desirable, to consider too realistic a model with «small billard balls plying between the interacting particles» (4). Ignoring this warning can lead to serious troubles in theoretical exposition. An obvious example is furnished by the observation that positive energy boson exchange automatically results in a net repulsion between the interacting particles. How then can one reconcile the model with the possibility of an attraction? The difficulty has been illustrated by Harney (5) in the context of graviton exchange because imaginary coupling constants must be introduced in the model.

Since Feynman, many authors have reconsider the quantum theory of boson exchange, extending the problem to the multi-boson case. The most recent references may be found in the papers by Sucher (6, 7). For related works, see also (8, 9).

To our knowledge, nobody has never considered the problem from the classical point of view. In spite of expected difficulties, we found valuable to model the boson exchange in a relativistically invariant way such that it became possible to at least attain correct asymptotic interaction laws. This has been done in this paper and we believe that the results are worthy of interest. Working in one dimension, we have accurately retreived, in case I, Coulomb's potential law with its characteristic asymptotic $1/r$ decrease. We found, however, a non singular mass-dependent behaviour near the origin which attracted our attention. In case II, we have found the analytic expression for the corresponding short-range potential, which, surprisingly, does not coincide with the Yukawa potential law even in the asymptotic regime. Various generalizations are considered in more than one dimension.

II-1. Long-range interaction in one dimension

II-1.1. The principle of long-range repulsion

We first consider the simple example of two point-like, equivalent sign, charged particles with equal rest masses, m. They are temporarily maintained at rest at a distance of $2x_0$ from each other. Their momentum, p_0, is therefore zero. Classical electrostatics considers that they experience a repulsive force whose value is given by Coulomb's law,

$$F = \frac{1}{4\pi\varepsilon_0} \frac{\varepsilon\varepsilon'}{(2x_0)^2}$$

This force is responsible for the accelerated motion of the particles once released. Another equivalent way of describing the situation is to consider that each particle carries one-half of the total initial positive potential energy, U_0, of the system. This energy is given by the classical law,

$$U_0 = \frac{1}{4\pi\varepsilon_0}\frac{\varepsilon\varepsilon'}{2x_0}$$

When they are released, each particle progressively converts its potential energy, $U_0/2$, into kinetic energy, therefore accelerating at a rate predicted by the laws of motion.

A third possibility, based on Feynmann's ideas, is now investigated. We first make the assumption that a positive potential energy, E, is effectively stored in each particle initially at rest in the form of an excess mass $\delta_0 = E/c^2$. We shall see later that E is approximatively equal to $U_0/2$. The situation, illustrated in the first line of Fig. 1, is valid at any time $t \leq t_0$.

A further assumption is that, when the particles are released at zero time, t_0, they instantaneously emit a photon, or a graviton, of such a frequency, ν_1, that they recover their original rest mass, m. Of course, they automatically experience a recoil with momentum, π_1, see Fig. 1. At time $t_1 > t_0$, both photons simultaneously hit the other particle and are instantaneously absorbed. This increases once more the momenta of the particles. A new cycle, emission plus absorption, can immediately restart. The particles are thus moving away with an acceleration that we may, in principle, determine accurately.

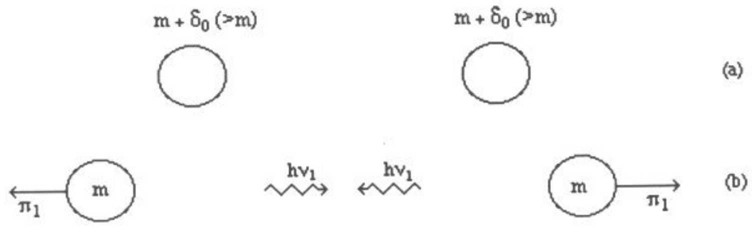

FIG. 1

Long range repulsion in one dimension.

II-1.2. The principle of long-range attraction

The line of argument developed in Section II-1.1 does not seem suitable to describe an attraction between particles because the boson exchanges generally entail a recoil of the interactive particles. However, the description of an attractive interaction becomes possible without altering the principle of the method, provided one considers that the virtual mediating bosons may be characterized by negative energies and momenta. For example, a negative energy photon of negative frequency, $\nu < 0$, is characterized by the negative energy, $W=h\nu$. Its momentum, $q=h\nu/c$, is directly opposite to its direction of motion, see Fig. 2.

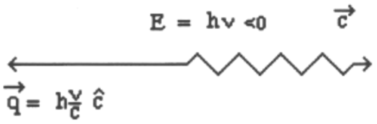

FIG. 2
The attributes of a negative energy photon.

Paraphrasing Section II-1.1, we again consider the simplified example of two attracting equal mass particles. They each initially present a negative mass defect, $\delta_0 < 0$, so that their initial effective masses are equal to $m+\delta_0$. When one releases the particles, they immediately emit a negative energy photon in the direction of the other particle of such a negative frequency, ν_1, that they instantaneously recover their original rest mass. It is immediately seen that the particles move together rather than apart, see Fig. 3. This is a consequence of the law of conservation of momentum.

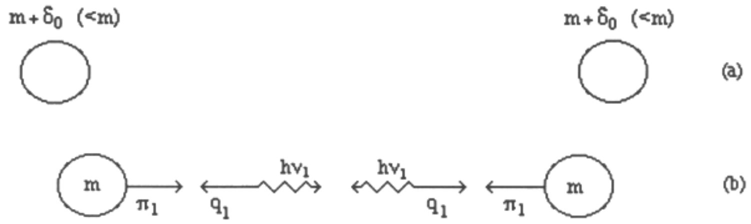

FIG. 3
Long range attraction in one dimension.

II-1.3. The equations of motion in the general one dimensional case

Neither Fig.1 nor Fig. 3 describes the most general case in which the particles are not initially at rest and their masses are unequal, say m and m', with $m' \geq m$. A prime will, from now on, indicate all the quantities associated with m'. We have thus modified Fig. 1 for the general case of unequal masses. This has been done in Fig. 4 which describes, in the zero momentum frame, a complete two-step repulsive cycle: emission + absorption + reemission + reabsorption. The reader will observe that we have set $t'_2=t_2$. This anticipates the simultaneity condition that will be established in Section II-1.5.

In order to save space, we have organized the calculations in such a way that both the repulsive and the attractive case are dealt with simultaneously. This requires a carefull discussion of the signs of the various involved quantities, scalars or vectors. The following settings seem reasonable; δ and ν are positive scalars in the repulsion problem and negative scalars otherwise and $\vec{p}$, $\vec{\pi}$, $\vec{r}$ are vectors. The reader must not confuse the p_x component of $\vec{p}$, which is an algebraic number, with the modulus, p, of the same vector which is always positive. For an obvious sake of simplicity, x will stand for r_x.

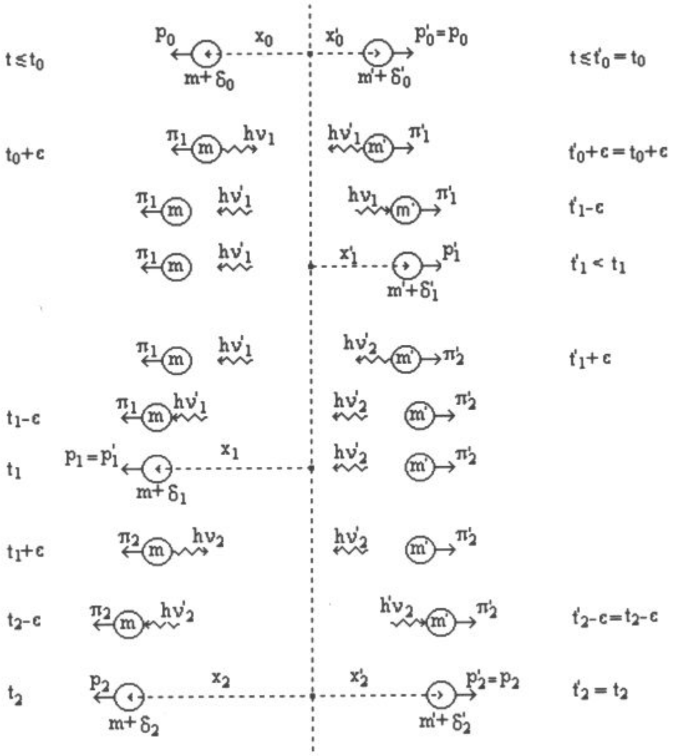

FIG. 4

Long range interaction in the general one-dimensional case.

The calculations are organized as follows. Working in the zero momentum frame, where $p'_{x,i}+p_{x,i} = 0$ for all i, each cycle is described by a set of recursive equations which express the conservation of energy and momentum and the propagation law of the photons. Recursive means that, at each step, the unknown quantities are calculated as functions of the same quantities calculated during the previous cycle. In order to lighten the notation, we choose natural units by setting $c=\hbar=1$ in the equations. Natural units will be used throughout this paper. Moreover $\hat{a}$ will systematically stand for the unitary vector parallel to $\vec{a}$. Notice that, in one dimension, there is no difference between $\pi_{x,i+1}{}^2$ and $\pi_{i+1}{}^2$ and that the same is true with $p_{x,i}{}^2$ and $p_i{}^2$.

A complete cycle is determined by two sets of equations (i = 0, 1, 2, …). Firstly an «energy-momentum» set,

[1a, b] $$\begin{cases} \left[(m+\delta_i)^2 + p_i^2\right]^{1/2} = \left[m^2 + \pi_{i+1}^2\right]^{1/2} + \nu_{i+1} \\ p_{x,i} = \pi_{x,i+1} + \nu_{i+1} \end{cases}$$

[2a, b] $$\begin{cases} \left[(m'+\delta'_i)^2 + p'^2_i\right]^{1/2} = \left[m'^2 + \pi'^2_{i+1}\right]^{1/2} + \nu'_{i+1} \\ p'_{x,i} = \pi'_{x,i+1} - \nu'_{i+1} \end{cases}$$

[3a, b] $$\begin{cases} \left(m^2 + \pi_{i+1}^2\right)^{1/2} + \nu'_{i+1} = \left[(m+\delta_{i+1})^2 + p_{i+1}^2\right]^{1/2} \\ p_{x,i+1} = \pi_{x,i+1} - \nu'_{i+1} \end{cases}$$

[4a, b] $$\begin{cases} \left(m'^2 + \pi'^2_{i+1}\right)^{1/2} + \nu_{i+1} = \left[(m' + \delta'_{i+1})^2 + p'^2_{i+1}\right]^{1/2} \\ p'_{x,i+1} = \pi'_{x,i+1} + \nu_{i+1} \end{cases}$$

and secondly a «space-time» set,

[5a, b] $$\begin{cases} x'_{i+1} - x'_i = v'_{x,i+1}(t'_{i+1} - t'_i) \\ x_{i+1} - x_i = v_{x,i+1}(t_{i+1} - t_i) \end{cases}$$

[6a, b] $$\begin{cases} x'_{i+1} = x_i + t'_{i+1} - t_i \\ x_{i+1} = x'_i - t_{i+1} + t'_i \end{cases}$$

where the velocities are related to the momenta by the following way,

[7a, b] $$\begin{cases} \vec{v}_i = \dfrac{\vec{\pi}_i}{(m^2 + \pi_i^2)^{1/2}} \\ \vec{v}'_i = \dfrac{\vec{\pi}'_i}{(m'^2 + \pi'^2_i)^{1/2}} \end{cases}$$

The solution of equations [1] to [7] is unique provided the following initial data, the masses m and m', $t_0=t'_0=0$, without loss of generality, and x_0 and x'_0 are given. Though the choice of the origin is unimportant it may be judicious to assimilate it with the correctly defined center of mass of the system. In this case we denote the initial positions by «cm» subscripts, $x_{cm,0}$ and $x'_{cm,0}$. Are also given, the initial momenta $p_{x,0}$ and $p'_{x,0}=-p_{x,0}$, which vanish if the particles are initially at rest and the initial mass deviations δ_0 and δ'_0. We expect that the sum of these quantities is nearly equivalent to the mutual potential energy of the particles in their initial state, $\delta_0 + \delta'_0 \sim U_0/c^2$. But to obtain the exact values of δ_0 and δ'_0, we need a conjecture that will be formulated in Section II-1.5.

II-1.4. The recursive solution of the equations of motion

For the sake of brevity we first define some useful quantities, (i = 0, 1, ...),

[8a, b] $z_i=\delta_i^2+2m\delta_i$ and $z'_i=\delta'^2_i+2m'\delta'_i$,

[9a, b] $W_i=[(m+\delta_i)^2+p_i^2]^{1/2}$ and $W'_i=[(m'+\delta'_i)^2+p'^2_i]^{1/2}$.

The recursive computation of the solution of equations [1] to [7] is written as,

Step one, fix initial values for: m, m', x_0, x'_0, t_0, t'_0, δ_0, δ'_0, $p_{x,0}$, $p'_{x,0}$. Calculate z_0, z'_0, W_0, W'_0.

Step two, for i = 0, 1, ..., iterate the following six sets of equations.

Set one, solve equations [1] and [2] under the form,

[10a, b]
$$\begin{cases} \nu_{i+1} = \dfrac{z_i}{2(W_i - p_{x,i})} \\ \nu'_{i+1} = \dfrac{z'_i}{2(W'_i + p'_{x,i})} \end{cases}$$

[11a, b]
$$\begin{cases} \pi_{x,i+1} = p_{x,i} - \nu_{i+1} \\ \pi'_{x,i+1} = p'_{x,i} + \nu'_{i+1} \end{cases}$$

Set two, evaluate the following square roots which appear to be rational expressions,

[12a, b]
$$\begin{cases} (m^2 + \pi^2_{x,i+1})^{1/2} = \dfrac{m^2 + (W_i - p_{x,i})^2}{2(W_i - p_{x,i})} \\ (m'^2 + \pi'^2_{x,i+1})^{1/2} = \dfrac{m'^2 + (W'_i - p'_{x,i})^2}{2(W'_i + p'_{x,i})} \end{cases}$$

Set three, solve equations [3b] and [4b] under the form,

[13a, b]
$$\begin{cases} p_{x,i+1} = \pi_{x,i+1} - \nu'_{i+1} \\ p'_{x,i+1} = \pi'_{x,i+1} + \nu_{i+1} \end{cases}$$

and verify that is constant for all i, zero in the zero momentum frame.

Set four, calculate the following expressions, using eventually equation [12]:

[14a, b]
$$\begin{cases} W_{i+1} = (m^2 + \pi^2_{x,i+1})^{1/2} + \nu'_{i+1} \\ W'_{i+1} = (m'^2 + \pi'^2_{x,i+1})^{1/2} + \nu_{i+1} \end{cases}$$

[15a, b]
$$\begin{cases} z_{i+1} = W_{i+1}^2 - m^2 - p_{x,i+1}^2 \\ z'_{i+1} = W'^2_{i+1} - m'^2 - p'^2_{x,i+1} \end{cases}$$

[16a, b]
$$\begin{cases} \delta_{i+1} = -m + (m^2 + z_{i+1})^{1/2} \\ \delta'_{i+1} = -m' + (m'^2 + z'_{i+1})^{1/2} \end{cases}$$

Set five, ensure that the energy is conserved under the form,

[17]
$$W_{i+1} + W'_{i+1} = W_0 + W'_0 = W$$

Set six, evaluate the velocities, the positions and the times from,

[18a, b]
$$\begin{cases} v_{x,i+1} = \pi_{x,i+1} / (m^2 + \pi_{i+1}^2)^{1/2} \\ v'_{x,i+1} = \pi'_{x,i+1} / (m'^2 + \pi'^2_{i+1})^{1/2} \end{cases}$$

[19a, b]
$$\begin{cases} x'_{i+1} = [x'_i - v'_{x,i+1}(x_i + t'_i - t_i)]/(1 - v'_{x,i+1}) \\ x_{i+1} = [x_i + v_{x,i+1}(x'_i + t'_i - t_i)]/(1 + v_{x,i+1}) \end{cases}$$

[20a, b]
$$\begin{cases} t'_{i+1} = x'_{i+1} - x_i + t_i \\ t_{i+1} = x'_i + t'_i - x_{i+1} \end{cases}$$

Step three, display the results in the desired way. Examples will be Figs. 5 and 6.

II-1.5. The condition of simultaneity

The recursive scheme of Section II-1.4 can only start once the global initial mass deviation, defined in Section II-1.3, is correctly shared between the particles under the form δ_0 and δ'_0. The problem was immaterial in the simplified example because the equality between the masses of the particles implied, by symmetry, the equipartition $\delta_0 = \delta'_0$. But in the general case of unequal masses we need an additional conjecture.

In fact, if $m'>m$ one has that the recoil velocity, v'_1, of m' is less than that, v_1, of m. The consequence is that m' will be hit by the photon earlier than m, in short: $t'_1<t_1$. Now we understand that m' will reemit its photon earlier than m so that it is not clear wether we will have t'_2 less than, equal, or greater than t_2. The most useful case would be that the equality holds, i.e., $t'_2 = t_2$. Precisely we conjecture that both photons must be reabsorbed simultaneously in the zero momentum frame. We shall show that this leads to a universal distribution law of the mass deviations δ_0 and δ'_0.

We postulate that the global initial mass deviation is automatically shared between the particles, under the form δ_0 and δ'_0, in such a way

that each two-step cycle, as described by Fig. 4, starts and ends simultaneously for both particles in the zero momentum frame.

That this simultaneity can hold is not at all evident and its demonstration requires an in-depth discussion of equations [1] to [7]. Solving analytically such a system of fourteen non-linear equations seems a rather formidable task, however, we have found that Wolfram's Mathematica (1) can help in solving them. The equations must be prepared before coding them in the Mathematica language such that a minimum number of square roots are calculated in order to shorten the calculation time, to save memory space, and to avoid sign indeterminacies. The Mathematica programs are presented hereafter in bold characters with the results following them. We have used obvious notations such that, pix[i] stands for $\pi_{x,i}$, piprx[i] for $\pi'_{x,i}$ and so on.

The following program writes the space-time equations [5] and [6] with i replaced by $2i$ and $2i+1$ successively. It then solves the resulting system for x_{2i+2}, x'_{2i+2}, t_{2i+2} and t'_{2i+2} in terms of the same 2i-indiced variables. It looks then for the necessary condition that $t'_{2i}=t_{2i}$ entails $t'_{2i+2}=t_{2i+2}$.

```
pgm1=Solve[{x[2i+1]==x[2i]+vx[2i+1](t[2i+1]-t[2i]),
xpr[2i+1]==xpr[2i]+vprx[2i+1](tpr[2i+1]-tpr[2i]),
x[2i+1]==xpr[2i]-(t[2i+1]-tpr[2i]),xpr[2i+1]==x[2i]+(tpr[2i+1]-t[2i]),
x[2i+2]==x[2i+1]+vx[2i+2](t[2i+2]-t[2i+1]),
xpr[2i+2]==xpr[2i+1]+vprx[2i+2](tpr[2i+2]-tpr[2i+1]),
x[2i+2]==xpr[2i+1]-(t[2i+2]-tpr[2i+1]),xpr[2i+2]==x[2i+1]+(tpr[2i+2]-t[2i+1])},
{x[2i+2],xpr[2i+2],t[2i+2],tpr[2i+2]},{x[2i+1],xpr[2i+1],t[2i+1],tpr[2i+1]}]
Factor[(tpr[2i+2]/. pgm1)-(t[2i+2]/. pgm1)/.(tpr[2i]->t[2i])]
```

The program outputs the resulting condition, rewritten as:

$$[21] \qquad \frac{v_{x,2i+2}v'_{x,2i+2}+1}{v_{x,2i+2}+v'_{x,2i+2}}=\frac{v_{x,2i+1}v'_{x,2i+1}+1}{v_{x,2i+1}+v'_{x,2i+1}}$$

We now transform equation [21] to obtain the fundamental sharing law of the mass deviations.

Theorem 1. The simultaneity condition is possible, in the zero momentum frame, if and only if at each even step $2i$, the mass deviations obey the universal sharing law,

[22] $\delta_{2i}(2m+\delta_{2i})=\delta'_{2i}(2m'+\delta'_{2i})$, or in shorter notation, $z_{2i}=z'_{2i}$.

Proof. The following program encodes the recursive solution of Section II-1.4 for arbitrary initial conditions and it looks for the condition that equation [21] will be satisfied.

```
pprx[2i]=-px[2i];nu[2i+1]=z[2i]/(2(W[2i]-px[2i]));
nupr[2i+1]=zpr[2i]/(2(Wpr[2i]+pprx[2i]));
pix[2i+1]=px[2i]-nu[2i+1];piprx[2i+1]=pprx[2i]+nupr[2i+1];
rad[2i+1]=(m^2+(W[2i]-px[2i])^2)/(2(W[2i]-px[2i]));
radpr[2i+1]=(mpr^2+(Wpr[2i]+pprx[2i])^2)/(2(Wpr[2i]+pprx[2i]));
vx[2i+1]=Simplify[Together[pix[2i+1]/rad[2i+1]]] /.z[2i]->W[2i]^2-px[2i]^2-m^2;
vprx[2i+1]=Simplify[Together[piprx[2i+1]/radpr[2i+1]]]
                                    /.zpr[2i]->Wpr[2i]^2-pprx[2i]^2-mpr^2;
vx[2i+2]=(m^2-px[2*i+1]^2+2*px[2*i+1]*W[2*i+1]-W[2*i+1]^2)/
(m^2+px[2*i+1]^2-2*px[2*i+1]*W[2*i+1]+W[2*i+1]^2);
vprx[2i+2]=(-mpr^2+px[2*i+1]^2-2*px[2*i+1]*Wpr[2*i+1]+Wpr[2*i+1]^2)/
(mpr^2+px[2*i+1]^2-2*px[2*i+1]*Wpr[2*i+1]+Wpr[2*i+1]^2);
px[2i+1]=pix[2i+1]-nupr[2i+1];pprx[2i+1]=piprx[2i+1]+nu[2i+1];
W[2i+1]=nupr[2i+1]+rad[2i+1];Wpr[2i+1]=nu[2i+1]+radpr[2i+1];
test=Numerator[Together[(vx[2i+2]vprx[2i+2]+1)(vprx[2i+1]+vx[2i+1])]]-
Numerator[Together[(vx[2i+1]vprx[2i+1]+1)(vprx[2i+2]+vx[2i+2])]];
testprime=Factor[Simplify[Expand[test]]];
condition=Simplify[Expand[Numerator[Together[testprime]]/.
{z[2i]->W[2i]^2-m^2-px[2i]^2,zpr[2i]->Wpr[2i]^2-mpr^2-pprx[2i]^2}]]
```

The output result furnishes the searched condition,

```
128*m^2*mpr^2*(-px[2*i]+W[2*i])^4*(-px[2*i]+Wpr[2*i])^4*
(-m^2+mpr^2+W[2*i]^2-Wpr[2*i]^2)*(m^2+mpr^2+2*px[2*i]*W[2*i]-
 W[2*i]^2+2*px[2*i]*Wpr[2*i]-2*W[2*i]*Wpr[2*i]-Wpr[2*i]^2)=0
```

Looking at the order of magnitude of the various terms, one finds that only one factor may vanish, i.e., $m'^2-m^2+W_{2i}^2-W'^2_{2i}$, a result which is equivalent to condition [22]. That condition seems desperately too severe to be fulfilled. Fortunately however we have the following theorem.

Theorem 2. If the simultaneity condition, $\delta_{2i}(2m+\delta_{2i})=\delta'_{2i}(2m'+\delta'_{2i})$, holds at time $i=0$ then it will hold at each later even time, $2i$.

Proof. Eliminating the p_x and the p'_x between equations [1] to [4] leaves four equations,

$$4m^2\nu_{i+1}^2 - 4z_i\pi_{x,i+1}\nu_{i+1} = z_i^2 \qquad [23]$$

$$4m'^2\nu'^2_{i+1} + 4z'_i\pi'_{x,i+1}\nu'_{i+1} = z'^2_i \qquad [24]$$

[25] $$4m^2\nu'^2_{i+1} + 4z_{i+1}\pi_{x,i+1}\nu'_{i+1} = z^2_{i+1}$$

[26] $$4m'^2\nu^2_{i+1} - 4z'_{i+1}\pi'_{x,i+1}\nu_{i+1} = z'^2_{i+1}$$

The following program discusses their compatibility;

```
Reduce[{Eliminate[{4m^2 nu[i+1]^2-z[i]^2-4 z[i]pix[i+1]nu[i+1]==0,
4m^2 nupr[i+1]^2-z[i+1]^2+4 z[i+1]pix[i+1]nupr[i+1]==0},pi[i+1]],
Eliminate[{4mpr^2 nupr[i+1]^2-zpr[i]^2+4 zpr[i]piprx[i+1]nupr[i+1]==0,
4mpr^2 nu[i+1]^2-zpr[i+1]^2-4 zpr[i+1]piprx[i+1]nu[i+1]==0},pipr[i+1]]},
{nu[i+1],nupr[i+1],z[i+1],zpr[i+1]}]
```

One finds that the following relations must hold for all i,

[27] $$m'^2 z_i z_{i+1} = m^2 z'_i z'_{i+1}$$

and

[28] $$4m'^2 \nu_{i+1}\nu'_{i+1} = z'_i z'_{i+1}$$

We now concentrate on equation [27] which is a first order recurrence which may be solved exactly. It suffices to rewrite it under the equivalent form,

$$(m/m')(z'_{i+1}/z_{i+1})=1/[(m/m')(z'_i/z_i)].$$

Its solution is immediate:

$$\begin{cases} z_{2i}/z'_{2i} = z_0/z'_0 \\ (m'^2 z_{2i+1})/(m^2 z'_{2i+1}) = z'_0/z_0 \end{cases}$$

It now becomes evident that the equality, $z'_0=z_0$, entails:

[29] $$\begin{cases} z_{2i} = z'_{2i} \\ m'^2 z_{2i+1} = m^2 z'_{2i+1} \end{cases}$$

and this achieves the proof of theorem 2.

In conclusion, working in the zero momentum frame, the simultaneity condition is fulfilled, in the one-dimensional motion, if one respects the sharing law, $z'_0=z_0$, or equivalently, $\delta_0(2m+\delta_0) = \delta'_0(2m'+\delta'_0)$, of the total initial mass deviation of the system.

II-1.6. The numerical solution of the equations of motion

The interaction of two unequal masses, governed by equation [1] to [7] may be considered as solved if the abscissas x_i and x'_i of both masses can be displayed as functions of the times t_i. This can be achieved numerically by applying the recursive scheme of Section II-1.4 to a set of initial conditions. The total initial mass deviation, $\Delta_0=\delta_0+\delta'_0$, must be shared between the two particles in agreement with $z'_0=z_0$. This leads to the following initialization,

$$[30] \qquad \delta_0 = \frac{\Delta_0(\Delta_0+2m')}{2(\Delta_0+m+m')} \qquad \text{and} \qquad \delta'_0 = \frac{\Delta_0(\Delta_0+2m)}{2(\Delta_0+m+m')}$$

The value of Δ_0 is given and it depends on the strength of the coupling at the starting point. Its sign is the same as that of δ_0 and δ'_0. In the following example, we treat the repulsive case.

Working in the zero momentum frame, we have started at the initial time $t_0=t'_0=0$; x_0 and x'_0 are given. Though this is not essential, we shall see later that the origin of the frame coincides with the center of mass of the system if one respects the relation,

$$[31] \qquad x_{cm,0}[(m+\delta_0)^2+p_0^2]^{1/2} = -x'_{cm,0}[(m'+\delta'_0)^2+p_0^2]^{1/2}$$

The numerical solution of equations [1] to [7] is obtained by the following program where the initial conditions are accurate to 25 significant figures,

```
m=1;mpr=2;d[0]=1/100;dpr[0]=-mpr+N[Sqrt[mpr^2+d[0]^2+2m d[0]],25];
z[0]=d[0]^2+2m*d[0];zpr[0]=dpr[0]^2+2mpr*dpr[0];px[0]=2/100;pprx[0]=-2/100;
En=N[Sqrt[(m+d[0])^2+px[0]^2]+Sqrt[(mpr+dpr[0])^2+px[0]^2],25];
W[0]=N[(En^2-mpr^2+m^2)/(2En),25];Wpr[0]=N[(En^2+mpr^2-m^2)/(2En),25];
t[0]=0;tpr[0]=0;x[0]=-1;xpr[0]=2;
xcm[0]=N[(x[0]-xpr[0])Wpr[0]/En,25];xprcm[0]=N[(xpr[0]-x[0])W[0]/En,25];
Do[{{nu[i+1]=z[i]/(2(W[i]-px[i])),nupr[i+1]=zpr[i]/(2(Wpr[i]+pprx[i]))},
{pix[i+1]=px[i]-nu[i+1],piprx[i+1]=pprx[i]+nupr[i+1]},
{rad[i+1]=(m^2+(W[i]-px[i])^2)/(2(W[i]-px[i])),
radpr[i+1]=(mpr^2+(Wpr[i]-px[i])^2)/(2(Wpr[i]+pprx[i]))},
{px[i+1]=pix[i+1]-nupr[i+1],pprx[i+1]=piprx[i+1]+nu[i+1]},
{W[i+1]=rad[i+1]+nupr[i+1],Wpr[i+1]=radpr[i+1]+nu[i+1]},
{z[i+1]=W[i+1]^2-m^2-px[i+1]^2,zpr[i+1]=Wpr[i+1]^2-mpr^2-pprx[i+1]^2},
{d[i+1]=-m+(m^2+z[i+1])^(1/2),dpr[i+1]=-mpr+(mpr^2+zpr[i+1])^(1/2)},
{vx[i+1]=pix[i+1]/rad[i+1],vprx[i+1]=piprx[i+1]/radpr[i+1]},
{xpr[i+1]=(xpr[i]-vprx[i+1]*(x[i]+tpr[i]-t[i]))/(1-vprx[i+1]),x[i+1]=(x[i]+vx[i+1]*
(xpr[i]+tpr[i]-t[i]))/(1+vx[i+1])},If[Mod[i+1,2]==0,{xcm[i+1]=(x[i+1]-xpr[i+1])*
```

```
Wpr[i+1]/En,xprcm[i+1]=(xpr[i+1]-x[i+1])W[i+1]/En},a=0],
{tpr[i+1]=xpr[i+1]-x[i]+t[i],t[i+1]=xpr[i]-x[i+1]+tpr[i]}}, {i,0,N}]
```

The numerical results may be displayed in any desired way. Because we are not mainly interested in the details of the trajectory of the particles, we first concentrate on the x-dependence of the interaction law. Two approaches are in fact possible.

A first approach is based on the existence of a rather obvious invariant of equation [1] to [4] i.e.,

$$\left([m^2+\pi_{i+1}^2]^{1/2}+[m'^2+\pi'^2_{i+1}]^{1/2}\right)+\left(\nu_{i+1}+\nu'_{i+1}\right)=W \qquad [32]$$

Because the terms in the first parenthesis represent the total kinetic energy of the system, the second parenthesis can be viewed as the total potential energy, U_i. We may display the graph of the time evolution of U_i, as shown in Fig. 5.

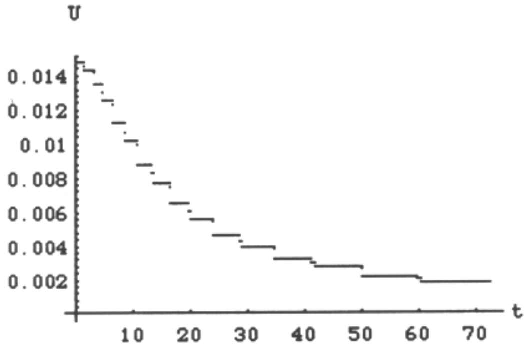

FIG. 5

The 1/r decrease of the long range repulsion potential in one dimension.

A similar graph can be displayed with U represented as a function of the distance between the particles. It is immediately apparent that this approach is not too pleasing because of the step-like character of the graph with repeated discontinuities at the successive times t'_i and t_i. We mention that it is possible to interpolate the graph by a smooth function which would exhibit the expected $1/x$ dependence. However because of this interpolation, and thus its rather arbitrary character, we shall abandon this approach to concentrate on a much more interesting second approach.

Because the initial state of the system was described in term of a mass deviation, $\Delta_0 = \delta_0 + \delta'_0$, adequately shared between the particles, it seems natural to follow its evolution, $\Delta_{2i} = \delta_{2i} + \delta'_{2i}$, at times t_{2i}. Because the following quantity is also an invariant of the system i.e.,

$$[33] \qquad [(m+\delta_i)^2 + p_i^2]^{1/2} + [(m'+\delta_i')^2 + p_i^2]^{1/2} = W_i + W_i' = W$$

it is immediately seen, by comparison with equation (32), that Δ_{2i}/c^2 is very close to the total usual potential energy. Fig. 6 presents the evolution of Δ_{2i} as a function of the mutual distance, $r_{2i} = |x'_{2i} - x_{2i}|$. The shape of the curve is consistent with the classical $1/r$ asymptotically decreasing law. That point will be discussed in detail in Section II-1.8.

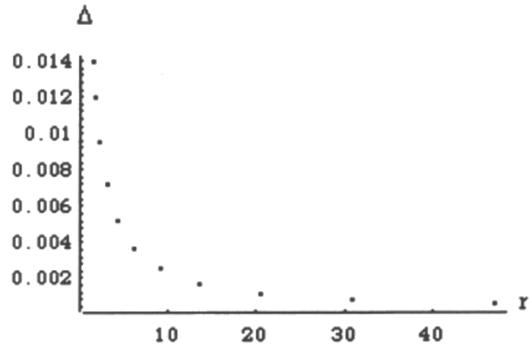

FIG. 6

The decreasing law of the mass deviation.

II-1.7. The analytical solution of the equations of motion

This is a much more delicate problem that must be solved in several steps.

Step one: a recurrence relation between the three successive mass deviations of a complete two-step cycle.

Theorem 3. If the simultaneity condition [22] is fulfilled, the following relation holds between successive z values,

$$[34] \qquad m^2 m'^2 z_{2i+1}^2 = z_{2i} z_{2i+2} (m^2 + z_{2i+1})^2$$

As a consequence of equation [29], an analogous relation exists between the z' values,

$$[35] \qquad m^2 m'^2 z'^2_{2i+1} = z'_{2i} z'_{2i+2} (m'^2 + z'_{2i+1})^2$$

Proof. The following program replaces i by $2i$ in equations [23], [26], [27], [28], [3b] and [4b] and then eliminates ν_{2i+1}, $\pi_{x,2i+1}$ and $\pi'_{x,2i+1}$.

```
elim1=Eliminate[{4m^2 nu[2i+1]^2-4 z[2i]pix[2i+1]nu[2i+1]==z[2i]^2,
4m^4 nu[2i+1]^2-4m^2 z[2i+1]piprx[2i+1]nu[2i+1]==mpr^2 z[2i+1]^2,
pix[2i+1]-z[2i]z[2i+1]/(4m^2 nu[2i+1])==px[2i+1],
piprx[2i+1]+nu[2i+1]==-px[2i+1]},{nu[2i+1],pix[2i+1],piprx[2i+1]}]
```

The result is a rational expression for $p^2_{x,2i+1}$ in terms of z_{2i} and z_{2i+1}. We do not give details here for the sake of brei+vity. An analogous program replaces i by $2i$ in equations [23], [26], [27], [28], [1b] and [2b] and it eliminates ν_{2i+2}, $\pi_{x,2i+2}$ and $\pi'_{x,2i+2}$:

```
elim2=Eliminate[{4m^2 nu[2i+2]^2-4 z[2i+1]pix[2i+2]nu[2i+2]==z[2i+1]^2,
4mpr^2 nu[2i+2]^2-4z[2i+2]piprx[2i+2]nu[2i+2]==z[2i+2]^2,
z[2i+1]z[2i+2]/(4m^2 nu[2i+2])-piprx[2i+2]==px[2i+1],
px[2i+1]==pix[2i+2]+nu[2i+2]},{nu[2i+2],pix[2i+2],piprx[2i+2]}]
```

The result is another rational expression for the same $p^2_{x,2i+1}$ in terms of z_{2i+1} and z_{2i+2}. Equating these two expressions furnishes the following factorized equation,

```
(-z[2*i]+z[2+2*i])*
(m^2*mpr^2*z[1+2*i]^2-m^4*z[2i]*z[2+2i]-2*m^2*z[2*i]*z[1+2*i]*z[2+2*i]-
                                        z[2*i]*z[1+2*i]^2*z[2+2*i])*
(m^4*z[2*i]*z[1+2*i]+m^2*mpr^2*z[1+2*i]^2+m^2*z[2*i]*z[1+2*i]^2+m^4*z[2*i]*z[2+2*i]+
m^4*z[1+2*i]*z[2+2*i]^2*m^2*z[2*i]*z[1+2*i]*z[2+2*i]+m^2*z[1+2*i]^2*z[2+2*i]
                                        +z[2*i]*z[1+2*i]^2*z[2+2*i])*
(m^2*mpr^2*z[2*i]*z[1+2*i]+m^2*mpr^2*z[1+2*i]^2+mpr^2*z[2*i]*z[1+2*i]^2+
m^4*z[2*i]*z[2+2*i]+m^2*mpr^2*z[1+2*i]*z[2+2*i]+2*m^2*z[2*i]*z[1+2*i]*z[2+2*i]+
                  mpr^2*z[1+2*i]^2*z[2+2*i]+z[2*i]*z[1+2*i]^2*z[2+2*i])=0
```

If one observes that each z-factor is of the order of magnitude of 2mδ, it is immediately seen that the two last factors never vanish. The first factor could only vanish if one had $z_{2i+2}=z_{2i}$, a result which is obviously unphysical. The conclusion is that the second factor must be zero, a result which proves theorem 3.

Equation [34] may be inverted under the useful form:

$$z_{2i+1} = m^2(z_{2i}z_{2i+2})^{1/2}/[mm'-(z_{2i}z_{2i+2})^{1/2}] \qquad [36]$$

Step two: rational expressions for the velocities as functions of the z values.

One has the following results,

$$v_{x,i+1} = \frac{m^2 z_{i+1}z'_{i+1} - z_i^2 z_{i+1} + z_i z_{i+1}^2 - m^2 z_i z'_i}{m^2 z_i z'_i + m^2 z_{i+1} z'_{i+1} + 2m^2 z_i z_{i+1} + z_i^2 z_{i+1} + z_i z_{i+1}^2} \qquad [37]$$

$$[38] \qquad v'_{x,i+1} = \frac{m'^2 z_i z'_i - m'^2 z_{i+1} z'_{i+1} + z'^2_i z'_{i+1} - z'_i z'^2_{i+1}}{m'^2 z_i z'_i + m'^2 z_{i+1} z'_{i+1} + 2m'^2 z'_i z'_{i+1} + z'^2_i z'_{i+1} + z'_i z'^2_{i+1}}.$$

Proof.

The acceptable solution of equations [23] to [28] in terms of the z_i and z'_i is calculated by the following program. Although four solutions are found only the second one furnishes frequencies ν_{i+1} and ν'_{i+1} of the correct sign.

```
part1={Solve[{(-4m^2 nu[i+1]^2+z[i]^2)/(4 z[i]nu[i+1])-nu[i+1]==
(-4mpr^2 nupr[i+1]^2+zpr[i]^2)/(4 zpr[i]nupr[i+1])-nupr[i+1],
4mpr^2 nu[i+1]nupr[i+1]==zpr[i] zpr[i+1],
mpr^2 z[i]z[i+1]==m^2 zpr[i]zpr[i+1]},{nu[i+1],nupr[i+1]}],
nu[i+1]=nu[i+1]/.part1[[2]],
nupr[i+1]=nupr[i+1]/.part1[[2]],
pix[i+1]=Simplify[(-4m^2 nu[i+1]^2+z[i]^2)/(4 z[i]nu[i+1])],
piprx[i+1]=Simplify[(-4mpr^2 nupr[i+1]^2+zpr[i]^2)/(4 zpr[i]nupr[i+1])]}
```

For the sake of brevity we do not detail the results which are directly used in the following. Equations [7a,b] then allow one to compute the velocities. The following program, entitled «part2», uses the output results of program «part1» to calculate the square of the velocities,

```
part2:={vxsq[i+1]=Simplify[pix[i+1]^2/(m^2+pix[i+1]^2)]/.
m^2 zpr[i]zpr[i+1] ->mpr^2 z[i]z[i+1],
vprxsq[i+1]=Simplify[piprx[i+1]^2/(mpr^2+piprx[i+1]^2)]/.
mpr^2 z[i]z[i+1]->m^2 zpr[i]zpr[i+1]}
```

The outputs are perfect squares; extracting the root furnishes equations [37] and [38]. The sign indeterminacies are resolved by observing that v_x and v'_x must have the same sign as the dominant term in the numerators noting that the denominators are always positive. For example, if $z_{i+1}z'_{i+1}>z_i z'_i$, then the particles are moving inward so that one has effectively, $v_x>0$ and $v'_x<0$.

Step three: more about constants of the motion in the zero momentum frame.

Energy and momentum are of course conserved during each cycle. Arbitrary combinations of those constants are also invariant. Some of them, only valid in the zero momentum frame, will prove useful. Energy is conserved in every reference frames and we may write with the notations of equation [9],

$$W_i + W'_i = W$$

More can be said in the zero momentum frame because $p_i^2 = p_i'^2$. Eliminating p_i^2 between equations [9a, b], one finds,

$$W_i'^2 - W_i^2 = m'^2 + z_i' - m^2 - z_i$$

We immediately deduce the simple expressions for the individual energies,

$$\begin{cases} W_i = (W^2 - m'^2 + m^2 - z_i' + z_i)/(2W) \\ W_i' = (W^2 + m'^2 - m^2 + z_i' - z_i)/(2W) \end{cases}$$

If we assume that the simultaneity condition (29) is fulfilled, we find that the individual energies, W_{2i} and W'_{2i} are invariant at the even times t_{2i},

$$\begin{cases} W_{2i} = (W^2 - m'^2 + m^2)/(2W) = w \\ W_{2i}' = (W^2 + m'^2 - m^2)/(2W) = w' \end{cases}$$

Let us see how it is possible to express these constants, or any combinations of them, in terms of the z_i values. The example of $-w/w'$ will prove to be especially interesting for our purpose.

Let us first rewrite equations [23)], [26], [1b] and [2b] where i is replaced by 2i. Taking relations [27] to [29] into account leaves a system of four independent equations for the four unknowns, $\pi'_{x,2i+1}$, $\pi_{x,2i+1}$, ν_{2i+1}, z_{2i+1}.

$$\begin{cases} 4m^2\nu_{2i+1}^2 - 4z_{2i}\pi_{x,2i+1}\nu_{2i+1} = z_{2i}^2 \\ 4m^4\nu_{2i+1}^2 - 4m^2 z_{2i+1}\pi'_{x,2i+1}\nu_{2i+1} = m'^2 z_{2i+1}^2 \\ \pi_{x,2i+1} + \nu_{2i+1} + \pi'_{x,2i+1} - z_{2i}z_{2i+1}/(4m^2\nu_{2i+1}) = 0 \\ \pi_{x,2i+1} + \nu_{2i+1} = p_{x,2i} \end{cases}$$

The solution is easily found in terms of the data, p_{2i} and z_{2i},

$$[39] \quad \begin{cases} \nu_{2i+1} = (w + p_{x,2i})z_{2i}/[2(m^2 + z_{2i})] \\ \pi_{x,2i+1} = p_{x,2i} - \nu_{2i+1} \\ z_{2i+1} = m^2(w + p_{x,2i})(w' + p_{x,2i})z_{2i}/[(m^2 + z_{2i})(m'^2 + z_{2i})] \\ \pi'_{x,2i+1} = -p_{x,2i} + (w' + p_{x,2i})z_{2i}/[2(m'^2 + z_{2i})] \end{cases}$$

where we have set:

$$\begin{cases} w = (p_{x,2i}^2 + m^2 + z_{2i})^{1/2} \\ w' = (p_{x,2i}^2 + m'^2 + z_{2i})^{1/2} \end{cases}$$

Inverting the third relation of the set [39] allows one to find $p_{x,2i}$ in terms of the of the z_i values,

$$[40] \quad [p_{x,2i} - (m^2 + z_{2i} + p_{x,2i}^2)^{1/2}][p_{x,2i} - (m'^2 + z_{2i} + p_{x,2i}^2)^{1/2}] = m^2 z_{2i}/z_{2i+1}$$

The following program solves equation [40] for $p_{x,2i}$ and then calculates $(w/w')^2$,

```
Solve[(px-Sqrt[px^2+m^2+z[2i]])(px-Sqrt[px^2+mpr^2+z[2i]])== m^2 z[2i]/
                                                        z[2i+1],px];
Factor[Cancel[(px^2+ m^2+z[2i])/(px^2+ mpr^2+z[2i])]]
```

The output result is a perfect square. One deduces $-w/w'$ by extracting the proper root, retaining the minus sign because the result must be a negative constant, i.e. in detail,

```
-w/w' = -W2i/W'2i = (-m^4*z[2*i]^2-2*m^4*z[2*i]*z[1+2*i]-2*m^2*z[2*i]^2*z
[1+2*i]-m^2*mpr^2*z[1+2*i]^2-m^2*z[2*i]*z[1+2*i]^2-mpr^2*z[2*i]*z[1+2*i]^2-
z[2*i]^2*z[1+2*i]^2) / (m^4*z[2*i]^2+ 2*m^2*mpr^2*z[2*i]*z[1+2*i]+
2*m^2*z[2*i]^2*z[1+2*i]+m^2*mpr^2*z[1+2*i]^2+m^2*z[2*i]
        *z[1+2*i]^2+mpr^2*z[2*i]*z[1+2*i]^2+z[2*i]^2*z[1+2*i]^2)
```

Step four: definition of the center of mass.

Although the choice of the origin is unimportant in the calculations, we expect that the results will appear in their simplest form if one makes the origin coincide with the center of mass of the system, properly defined. In fact the appellation «center of energy» should be more appropriate.

Theorem 4. If the relation

$$[41] \qquad x_{cm,2i}[(m+\delta_{2i})^2+p_{2i}^2]^{1/2}=-x'_{cm,2i}[(m'+\delta'_{2i})^2+p_{2i}^2]^{1/2}$$

holds at time t_0, then it holds at every even times t_{2i}. It is equivalent to saying that the center of mass of the system remains at rest or, in other words, that the ratio, $x'_{cm,2i}/x_{cm,2i}=x'_{cm,0}/x_{cm,0}$, remains constant at the value: $-W_{2i}/W'_{2i} = -w/w'$.

Proof. We begin with the following program which substitutes successively $2i$ and $2i$+1 for i in the space-time equations [5], [6] and searches for the residual relations existing between the velocities and the positions defined at even times t_{2i} and t_{2i+2}.

```
Simplify[Eliminate[{xpr[2i+1]-xpr[2i]==vprx[2i+1](tpr[2i+1]-t[2i]),
x[2i+1]-x[2i]==vx[2i+1](t[2i+1]-t[2i]),
xpr[2i+2]-xpr[2i+1]==vprx[2i+2](t[2i+2]-tpr[2i+1]),
x[2i+2]-x[2i+1]==vx[2i+2](t[2i+2]-t[2i+1]),-x[2i]+xpr[2i+1]==tpr[2i+1]-t[2i],
xpr[2i]-x[2i+1]==t[2i+1]-t[2i],-x[2i+1]+xpr[2i+2]==t[2i+2]-t[2i+1],
xpr[2i+1]-x[2i+2]==t[2i+2]-tpr[2i+1]},{x[2i+1],t[2i],xpr[2i+1],t[2i+2],tpr[2i+1],
                                                        t[2i+1]}]]
```

Three independant relations are found between the non-eliminated variables. The first restores the simultaneity condition [21]. The two others are writen as,

[42a] $(1+v'_{x,2i+2})(x'_{2i}-v'_{x,2i+1}x_{2i})=(1-v'_{x,2i+1})(x'_{2i+2}+v'_{x,2i+2}x_{2i+2})$

[42b] $(1-v_{x,2i+2})(x_{2i}+v_{x,2i+1}x'_{2i})=(1+v_{x,2i+1})(x_{2i+2}-v_{x,2i+2}x'_{2i+2})$

Dividing equation (42a) by (42b), we obtain,

[43]
$$(1+v'_{x,2i+2})(x'_{2i}/x_{2i}-v'_{x,2i+1})(1+v_{x,2i+1})(1-v_{x,2i+2}x'_{2i+2}/x_{2i+2})=$$
$$(1-v'_{x,2i+1})(x'_{2i+2}/x_{2i+2}+v'_{x,2i+2})(1-v_{x,2i+2})(1+v_{x,2i+1}x'_{2i}/x_{2i})$$

We now prove that, if $x'_{cm,2i}/x_{cm,2i} = -W_{2i}/W'_{2i} = -w/w'$, then $x'_{cm,2i+2}/x_{cm,2i+2}$ automatically has the same value. This recursion will achieve the proof.

The value of the constant of the motion, $-W_{2i}/W'_{2i}$, has been calculated in step three in terms of the z_i values. The following program substitutes $x'_{cm,2i}/x_{cm,2i}$ by that result in equation [43]. Solving for $x'_{cm,2i+2}/x_{cm,2i+2}$, the program precisely retreives the same expression, $-w/w'$.

```
equ=Solve[(1+vprx[2i+2])(result-vprx[2i+1])(1+vx[2i+1])(1-y vx[2i+2])==
(1-vprx[2i+1])(y+vprx[2i+2])(1-vx[2i+2])(1+result vx[2i+1]),y];
y=Cancel[Together[y/.equ[[1]]/.z[2i+2]->m^2mpr^2 z[2i+1]^2/
                                    (z[2i](m^2+z[2i+1])^2)]]
```

This establishes the recurrence and terminates the proof.

In short the coordinates of the center of mass-energy are calculated as:

$$\begin{cases} x_{cm,2i}=(x_{2i}-x'_{2i})w'/W \\ x'_{cm,2i}=-(x_{2i}-x'_{2i})w/W \end{cases}$$

Step five: the spatial dependence law of the mass deviation.

Theorem 5. The following invariant quantities define the spatial dependence of the mass excesses,

[44a, b]
$$x_{cm,2i}(2m\delta_{2i}+\delta^2_{2i})=x_{cm,0}(2m\delta_0+\delta_0^2)=C^t$$
$$x'_{cm,2i}(2m'\delta'_{2i}+\delta'^2_{2i})=x'_{cm,0}(2m'\delta'_0+\delta'^2_0)=C^t$$

The proofs of [44a] and [44b] are quite similar. To prove [44a], we first rewrite equation [42a] in the special case where x_{2i} and x'_{2i} are replaced by $x_{cm,2i}$ and $x'_{cm,2i}$ respectively and we devide both members by $x_{cm,2i}$, while taking theorem 4 into account. The following program solves the remaining equation for $x_{cm,2i+2}$ / $x_{cm,2i}$. In this program, «resu» stands for the constant, $-w/w'$, calculated in step three.

```
equ=Solve[(1+vprx[2i+2])(resu-vprx[2i+1])==q(1-vprx[2i+1])(resu+vprx[2i+2]),q];
q=Cancel[Together[q/.equ[[1]]/.z[2i+2]->m^2mpr^2 z[2i+1]^2/
                                        (z[2i](m^2+z[2i+1])^2)]]
```

The output is z_{2i}/z_{2i+2}. We have thus proven the existence of the following recurrence which establishes theorem 5,

$$x_{\mathrm{cm},2i+2}z_{2i+2} = x_{\mathrm{cm},2i}z_{2i}.$$

Corollary. A corollary of theorem 5 is that relations similar to [44] may be written in terms of the mutual distance between the particles, independantly of the choice of any origin,

[45]
$$(x'_{2i}-x_{2i})(2m\delta_{2i}+\delta_{2i}^2)=(x'_0-x_0)(2m\delta_0+\delta_0^2)=C^t$$
$$(x'_{2i}-x_{2i})(2m'\delta'_{2i}+\delta'^2_{2i})=(x'_0-x_0)(2m'\delta'_0+\delta'^2_0)=C^t$$

II-1.8. Interpretation of the results

The consequences of theorems 4 and 5, and its corollary, are numerous.

1) The spatial dependences of δ_{2i} or δ'_{2i} are easily deduced from theorem 5:

[46]
$$\delta_{2i}=-m+[m^2+x_{cm,0}(2m\delta_0+\delta_0^2)/x_{cm,2i}]^{1/2}$$
$$\delta'_{2i}=-m'+[m'^2+x'_{cm,0}(2m\delta'_0+\delta'^2_0)/x'_{cm,2i}]^{1/2}$$

2) The total mass deviation, $\Delta_{2i} = \delta_{2i} + \delta'_{2i}$ (i=0,1,2,...), is very close to the mutual potential energy, U_{2i}, of the system. Its components are deduced from equation [30]:

[47]
$$\delta_{2i}=\frac{\Delta_{2i}(\Delta_{2i}+2m')}{2(\Delta_{2i}+m+m')} \quad \text{and} \quad \delta'_{2i}=\frac{\Delta_{2i}(\Delta_{2i}+2m)}{2(\Delta_{2i}+m+m')}$$

3) We obtain the spatial dependence of Δ_{2i} from that of δ_{2i} and δ'_{2i}. The final result has been rewritten as a function of the mutual distance, $r_{2i}=|x'_{2i}-x_{2i}|$, to give

[48]
$$\Delta_{2i}=-(m+m')+\left[m^2+\frac{r_0(2m\delta_0+\delta_0^2)}{r_{2i}}\right]^{1/2}+\left[m'^2+\frac{r_0(2m'\delta'_0+\delta'^2_0)}{r_{2i}}\right]^{1/2}$$

This is the function which interpolates exactly the dots of Fig. 6.

In summary, we have found, in one dimension, that the total mass deviation carried by the interacting electric charges, is given by the following general law, rewritten with usual units:

[49]
$$\Delta=-(m+m')c^2+\left[m^2c^4+\lambda/r\right]^{1/2}+\left[m'^2c^4+\lambda/r\right]^{1/2}$$

where:

$$\lambda=\frac{2mm'c^2}{m+m'}\frac{\varepsilon\varepsilon'}{4\pi\varepsilon_0}$$

The λ-value has been calculated so that, at large distances, we retrieve the expected 1/r behaviour of the standard Coulomb interaction law, $U_{st} = \varepsilon\varepsilon'/(4\pi\varepsilon_0 r)$. At short distances we observe that Δ becomes mass-dependent and moreover it behaves like $r^{-1/2}$.

In the special case of an attracting system, like a pair (e^-, e^+), we find a Δ-law, written as,

$$\Delta(e^-, e^-) = 2mc^2(-1 + [1 + \alpha(\hbar/mc)/r]^{1/2}),$$

where α is the fine structure constant. Note that the critical distance is as short as $\hbar/m_e c$.

The same could be said about the classical model of the hydrogen atom (p^+, e^-). One finds in that special case the Δ-law,

$$\Delta(p^+, e^-) = -m_e c^2\left[1 - \left(1 - \frac{2\alpha m_p}{m_p + m_e}\frac{(\hbar/m_e c)}{r}\right)^{1/2}\right] - m_p c^2\left[1 - \left(1 - \frac{2\alpha m_e}{m_p + m_e}\frac{(\hbar/m_p c)}{r}\right)^{1/2}\right]$$

and the critical distance is again of the order of $\hbar/m_e c$. A similar slight modification appears in the planetary problem. Considering the motion of a planet around the sun leads to the following interaction Δ-law,

$$\Delta(s, p) = -m_s c^2\left[1 - \left(1 - \frac{m_p^2}{m_s + m_p}\frac{2G}{c^2 r}\right)^{1/2}\right] - m_p c^2\left[1 - \left(1 - \frac{m_s^2}{m_s + m_p}\frac{2G}{c^2 r}\right)^{1/2}\right]$$

If the planet is the earth, for example, this modified law entails a 1% discrepancy from the 1/r law at a distance of about 300km from the center of the sun. The problem of possible cosmological consequences is left open.

II-1.9. Exact positions versus time.

Although we are not mainly interested in the complete writing of the trajectory let us however mention that positions, x_{2i} and x'_{2i}, and times t_{2i} may be written in closed algebraic form. This can be achieved in the following way.

Theorem 6. The quantities, $u_{2i} = \sqrt{z_{2i}}$, obey a second-order linear recurrence with constant coefficients,

$$u_{2i+4}^{-1} - (W^2 - m^2 - m'^2)/(mm')u_{2i+2}^{-1} + u_{2i}^{-1} = 0, \quad [50]$$

where W is defined as the constant total energy of the system, i.e.:

$$W = [m^2 + z_{2i} + p_{2i}^2]^{1/2} + [m'^2 + z_{2i} + p_{2i}^2]^{1/2}. \quad [51]$$

The theorem is proven by the null output of the following program,

```
inv1=(m^2*mpr^2*u[2*i]^2+2*m^3*mpr*u[2*i]*u[2+2*i]+m^2*mpr^2*u[2+2*i]^2
-m^2*u[2*i]^2*u[2+2*i]^2+mpr^2*u[2*i]^2*u[2+2*i]^2)/
(m^2*mpr^2*u[2*i]^2+2*m*mpr^3*u[2*i]*u[2+2*i]+m^2*mpr^2*u[2+2*i]^2
+m^2*u[2*i]^2*u[2+2*i]^2-mpr^2*u[2*i]^2*u[2+2*i]^2);
equ1=Solve[(x+Sqrt[x^2+a])(x+Sqrt[x^2+b])==c,x];x=x/.equ1[[2]];
Factor[Cancel[x^2+a]];Factor[Cancel[x^2+b]];
result=Cancel[((a*b+2*a*c+c^2)+(a*b+2*b*c+c^2))^2/(4c(a+c)(b+c))];
a=m^2+u[2i]^2;b=mpr^2+u[2i]^2;c=m^2 u[2i]^2/z[2i+1];
z[2i+1]=m^2 u[2i]u[2i+2]/(m mpr-u[2i]u[2i+2]);
equ2=Solve[inv1==c,u[2i+2]];root=u[2i+2]/.equ2[[2]];
inv2=inv1/.{2i->2i+2,2i+2->2i+4};
test=Numerator[Simplify[Together[inv1-inv2]]]/
(2m mpr(m^2-mpr^2)u[2i+2])/.(u[2i+4]->u[2i]u[2i+2]/(g u[2i]-u[2i+2]));
equ3=Solve[Numerator[Simplify[Expand[Together[test]]]]/
(u[2*i]*(g*u[2*i] - 2*u[2 + 2*i])*u[2 + 2*i])==0,g];g=g/.equ3[[1]];
Numerator[Simplify[Together[g+(m^2+mpr^2)/(m mpr)-result/(m mpr)]]]
output = 0
```

Because of equation [45], theorem 6 may be rewritten in terms of the square root of the successive distances, $x'_{2i} - x_{2i}$, namely,

$$[52]\quad \sqrt{x'_{2i+4}-x_{2i+4}}-(W^2-m^2-m'^2)/(mm')\sqrt{x'_{2i+2}-x_{2i+2}}+\sqrt{x'_{2i}-x_{2i}}=0$$

Similar relations exist with x'_{2i} and x_{2i} replaced by either $x_{cm,2i}$ or $x'_{cm,2i}$. This is a consequence of equation [44]. An equivalent recurrence which avoids the square roots also exists but it is non-homogenous. Recurrence [52] may be solved exactly. Equations [6a, b] furnish in turn linear recurrences which may also be solved for the times, t_{2i}, in closed form. We do not pursue the calculations which lead to cumbersome formulas which seem of little interest.

II-3 Extension of the model to higher dimensions: an application to the Kepler problem

In Sections II-1 and II-2, we have explicitely considered particles that experience rectilinear motion. Even if this it is not the case, the model remains valid in the zero momentum frame. If the initial momenta, $\vec{p}_0$, are opposite to each other but not directly opposite, then the motion becomes two dimensional. In this case the calculations are more complex though not intractable. As an example of a multidimensional problem, we consider the extension of the model to the Kepler problem. For the sake of simplicity, we assume that the attracting center is infinitely heavy and therefore at rest. The simplification in that case is that the infinitely heavy center never emits nor absorbs a boson. This is an immediate consequence of equation [30]. Only the moving particle repeatidly emits and absorbs the mediating boson which is regularly simply reflected by the center. Fig. 7 details the first steps in the evolution of the system. At time $t=t_0$, the particle experiences a negative mass defect, δ_0, which is very close to the mass equivalent of its attractive potential energy. Its momentum is $\vec{p}_0$ and its position, with respect to the attracting center, C, is $\vec{r}_0$. At the time $t_0+\varepsilon$, the particle emits an antiphoton of negative frequency ν_1 in the direction of the center which reflects it so that it collides with the particle at the later time, t_1, at its new position, $\vec{r}_1$. It is then instantaneously reabsorbed so that the particle experiences a modified negative mass defect, δ_1, and so on.

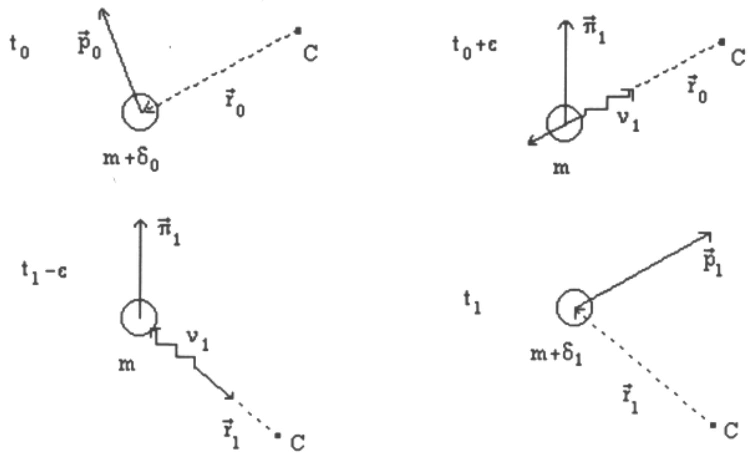

FIG. 7
The Keplerian interaction.

The evolution of the system is governed by the following sets of equations (i=0, 1, 2, ...),

[53a, b]
$$\begin{cases}\left[(m+\delta_i)^2+p_i^2\right]^{1/2}=\left[m^2+\pi_{i+1}^2\right]^{1/2}+\nu_{i+1}\\ \vec{\mathrm{p}}_i=\vec{\pi}_{i+1}-\nu_{i+1}\hat{r}_i\end{cases}$$

[54a, b]
$$\begin{cases}\left[m^2+\pi_{i+1}^2\right]^{1/2}+\nu_{i+1}=\left[(m+\delta_{i+1})^2+p_{i+1}^2\right]^{1/2}\\ \vec{p}_{i+1}=\vec{\pi}_{i+1}+\nu_{i+1}\hat{r}_{i+1}\end{cases}$$

[55a, b] $$\begin{cases} \vec{r}_{i+1} = \vec{r}_i + \vec{\pi}_{i+1}(t_{i+1} - t_i)/\left[(m^2 + \pi_{i+1}^2)\right]^{1/2} \\ r_i + r_{i+1} = t_{i+1} - t_i \end{cases}$$

where $\hat{a}$ denotes the unitary vector parallel to $\vec{a}$.

II-3.1. The analytical solution of the Kepler problem.

The recursive solution of the system requires that we extract the (i+1)-indiced variables as functions of the i-indiced variables. Using the following notation, $\vec{r}_i = (r_i\cos\varphi_i,\ r_i\sin\varphi_i)$, the resolution proceeds in five steps.

Step one: elementary constants of the motion.

A simple inspection of the equations of motion [53] to [55] reveals the existence of two obvious constants of the motion, i.e., the total energy, W, and the angular momentum, J. They may be written in various equivalent ways,

[56] $$\left[(m+\delta_i)^2 + p_i^2\right]^{1/2} = \left[m^2 + \pi_i^2\right]^{1/2} + \nu_i = W$$

[57] $$\vec{r}_i \times \vec{\pi}_i = \vec{\mathrm{r}}_i \times \vec{\pi}_{i+1} = \vec{r}_i \times \vec{p}_i = \vec{J}$$

Step two: some relations which may be deduced from [56] and [57].

Cross-multiplying equation [55a] by $\vec{r}_{i+1}$ and eliminating $(t_{i+1}-t_i)$ with the aid of equation [55b] leads to,

[58] $$r_i r_{i+1}\sin(\varphi_{i+1} - \varphi_i) = J(r_{i+1} + r_i)/\left[m^2 + \pi_{i+1}^2\right]^{1/2}$$

Squaring $\vec{r}_{i+1}-\vec{r}_i$ in equation [55a] and eliminating $(\mathrm{t}_{i+1}-\mathrm{t}_i)$ as previously furnishes:

[59] $$r_{i+1}^2 + r_i^2 - 2r_i r_{i+1}\cos(\varphi_{i+1} - \varphi_i) = (r_{i+1} + r_i)^2\pi_{i+1}^2/(m^2 + \pi_{i+1}^2)$$

Eliminating π_{i+1} between equations [58] and [59] gives

[60] $$r_i r_{i+1}\sin^2\left[(\varphi_{i+1} - \varphi_i)/2\right] = J^2/m^2$$

or, equivalently,

[61] $$r_i r_{i+1}\cos^2\left[(\varphi_{i+1} - \varphi_i)/2\right] = r_i r_{i+1} - J^2/m^2$$

Squaring equation [58] and eliminating $(\varphi_{i+1}-\varphi_i)$ with the aid of [60] gives:

[62] $$m^2 + \pi_{i+1}^2 = (m^2/4)(r_i + r_{i+1})^2/(r_i r_{i+1} - J^2/m^2)$$

Equations [60] and [62] permit a quick calculation of φ_{i+1} and π_{i+1} once r_{i+1} is known. We now calculate r_{i+1}.

Step three: a much less evident constant of the motion.

W and J are classical invariants of the Kepler system. Because our model introduces supplementary variables, like ν and π, it is not surprising that an additional constant of the motion exist. We have found that the following equivalent expressions are invariant for all i,

[63] $$2Wm(r_i r_{i+1})^{1/2}\cos[(\varphi_{i+1}-\varphi_i)/2]-m^2(r_{i+1}+r_i)=H$$

[64] $$2Wm\left[r_i r_{i+1}-J^2/m^2\right]^{1/2}-m^2(r_{i+1}+r_i)=H$$

[65] $$2m\nu_{i+1}\left[r_i r_{i+1}-J^2/m^2\right]^{1/2}=H$$

[66] $$2m\nu_{i+1}(r_i r_{i+1})^{1/2}cos[(\varphi_{i+1}-\varphi_i)/2]=H$$

The proof of [63] is intricate and we prefer to postpone its demonstration to Appendix A. Invariants [64] to [66] are more or less obvious consequences of [63]. Their proofs are therefore left to the reader.

Step four: explicit calculation and interpretation of the constant H.

Theorem 7. The expression,

[67] $$r_i(2m\delta_i+\delta_i^2) = H$$

is a constant of the two-dimensional motion and its value is also equal to H.

The proof of theorem 7 is postponed in Appendix B. The importance of the theorem may not be underestimated. It ensures that theorem 5 applies in more than one dimension. In particular the interaction law, (49), remains valid, in a simplified form because we have considered an infinitely heavy attracting center, i.e.,

[68] $$\Delta/c^2=-m+\left[m^2+H/r\right]^{1/2}$$

The constant H appears now related to the constant coupling strength of the center, i.e., $H=2m\varepsilon\varepsilon'/(4\pi\varepsilon_0 c^2)$, in the Coulomb problem and $H=-2GMm^2/c^2$, in the planetary problem.

The modified interaction law [68] deviates from the classical potential law when r ceases to be negligible in comparison with the critical distance, $r_{cr}=H/m^2$.

Step five: calculation of the trajectory.

Theorem 8. The successive radial positions, r_i, of the moving particle, obey a non-homogenous linear second-order recurrence relation with constant coefficients, i.e.;

[69] $$r_{i+1}+(2-4W^2/m^2)r_i+r_{i-1}+2H/m^2=0$$

Proof: one multiplies equation (A6) by $r_i^{1/2}$ and one applies equation [61] two times to obtain:

$$2Wr_i=m\left[r_{i-1}r_i-J^2/m^2\right]^{1/2}+m\left[r_ir_{i+1}-J^2/m^2\right]^{1/2}$$

Surprisingly, that non-linear recurrence may be linearized! It suffices to eliminate the square roots with the aid of equation [64] to obtain the desired result [69]. A recurrence like [69] is exactly soluble in terms of elementary functions. We conclude that the radial part of the trajectory is calculable in closed form. The resolution of [69] being a classical matter, we only give the final result,

[70] $$r_i=A\,sin(i\xi+\phi)-H/(2m^2-2W^2)$$

where ξ is determined by the relations,

$$\begin{cases} sin\xi=(2W/m^2)\left[m^2-W^2\right]^{1/2} \\ cos\xi=(2W^2-m^2)/m^2 \end{cases}.$$

Two initial conditions are thus necessary to fix the values of A and ϕ in [70]; r_0 is, of course, given and r_1 is deduced, for example, from equation [B3],

$$r_1=(r_0/m^2)\left[p_0^2+W^2+2W\vec{p}_0\cdot\hat{r}_0\right].$$

The times, t_i, may also be calculated exactly as solutions of recurrence [55b] and the angles φ_i follow from equation [60]. The trajectory may then be displayed in any desired way. An alternative method exists for obtaining the numerical solution of the Kepler problem. It suffices to eliminate π_{i+1} between equation [53a] and [53b]. One obtains,

$$\nu_{i+1}=\frac{\delta_i^2+2m\delta_i}{2(\hat{r}_i\cdot\vec{p}_i+[(m+\delta_i)^2+p_i^2]^{1/2})}$$

The other quantities easily follow from equations [53b] to [55b]. We have numerically tested this procedure, starting with the initial conditions, m=1, r_0=1, t_0=0, φ_0=0, p_0=0.02, δ_0=–0.001 and θ_0=angle($\vec{r}_0$,$\vec{p}_0$)=1.2radians.

```
m=1;r[0]={1,0};t[0]=0;p[0]=N[{Cos[6/5]/50,Sin[6/5]/50},25];
d[0]=N[-1/1000,25]; W=N[Sqrt[(m+d[0])^2+p[0].p[0]],25];
```

```
rmod[0]=Sqrt[r[0].r[0]];J=r[0][[1]]*p[0][[2]]-r[0][[2]]*p[0][[1]];
Do[{nu[i+1]=(d[i]^2+2m d[i])/(2(r[i].p[i]/rmod[i]+Sqrt[(m+d[i])^2+p[i].p[i]]));
pi[i+1]=p[i]+nu[i+1]r[i]/rmod[i];
rmod[i+1]=rmod[i](p[i].p[i]+W^2+2W p[i].r[i]/rmod[i])/m^2;
r[i+1]=r[i]+pi[i+1](rmod[i]+rmod[i+1])/Sqrt[m^2+pi[i+1].pi[i+1]];
t[i+1]=t[i]+rmod[i]+rmod[i+1];p[i+1]=pi[i+1]+nu[i+1]r[i+1]/rmod[i+1];
d[i+1]=-m+Sqrt[W^2-p[i+1].p[i+1]], {i,0,N}]
```

The results of the computations are displayed in Fig. 8. They show that the motion is quasi periodic as expected. We have verified that the rotation of the perihelion of the ellipse is consistent with the classical results obtained in the standard theory of special relativity because $r >> r_{cr}$ ($=2.10^{-3}$ in the example). It would increase if the initial conditions were drastically modified so that the particle enters in the domain where the potential law deviates from the $1/r$ behaviour.

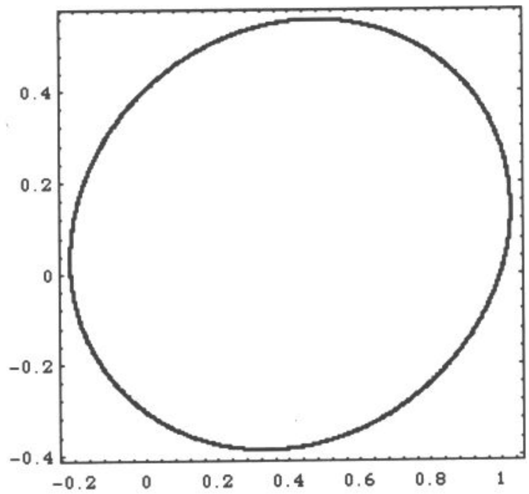

FIG. 8

An example of Keplerian trajectory.

III-1 Short-range interaction in one dimension

III-1.1. Short-range repulsion

In Sections II we have verified that the zero rest mass boson exchange theory is consistent with a long range interaction potential which decreases asymptotically according to the $1/r$ law. In this section, we extend the theory to finite rest mass boson exchange. We shall verify that such exchange results in a short range interaction and we will propose an analytical expression for the corresponding mass deviation. Fig. 9 illustrates, in the zero momentum frame, the short range repulsion between two particles of respective masses *m* and *m*'.

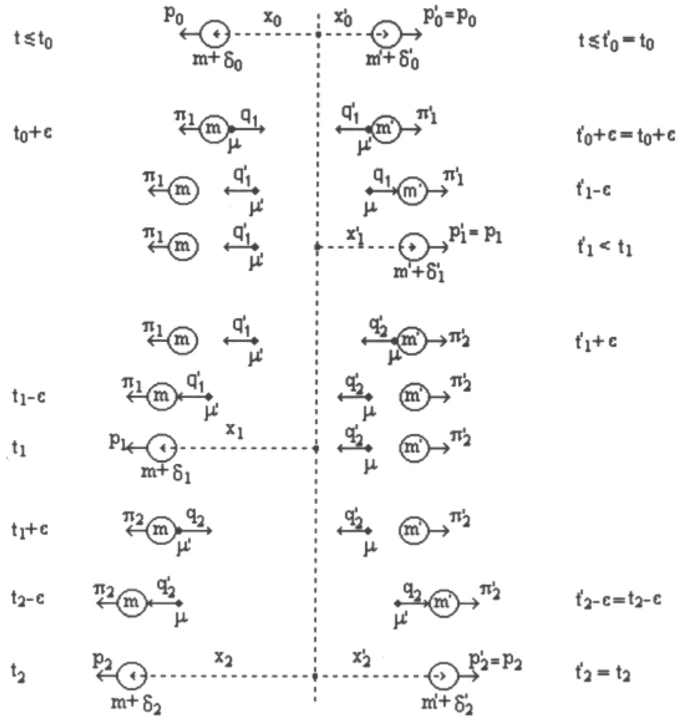

FIG.9

Short range interaction in the general one-dimensional case.

As in Section II-1 we assume that both particles exhibit small initial mass excesses, δ_0 and δ'_0 with δ_0 and $\delta'_0 > 0$. When the particles are released at time t_0, they each emit a boson of fixed rest mass, μ and μ', and they recoil, recovering immediately their respective rest masses, m and m'. The emitted bosons experience, thereafter, a sequence of elastic collisions with the particles which then accelerate in agreement with a law to be determined. It is interesting to note that, contrary to what happened in case I for photon exchange, the sequence of collisions is finite. Indeed, each collision between a particle and a boson decelerates it so that, after some time, the boson becomes too slow to once again catch the particle. From that time, the particles no longer accelerate but rather they pursue a rectilinear uniform motion. In other words, from a certain distance onwards, the interaction vanishes which means that it has a finite range. The same argument does not apply to case I because exchanged photons are progressively weakened in frequency, but not in velocity.

III-1.2. Short-range attraction.

Short-range attractions may be interpreted as the result of negative energy boson exchange. A negative energy boson of negative rest mass, $\mu<0$, which travels at the velocity, $\vec{w}$, see Fig. 10, is characterized by a negative energy, $W<0$, and by a momentum, $\vec{q}$, directly opposite to its velocity such that,

$$W = -\sqrt{\mu^2 c^4 + (cq)^2} \qquad \text{and} \qquad \vec{q} = \frac{\mu \vec{v}}{\sqrt{1 - v^2/c^2}}$$

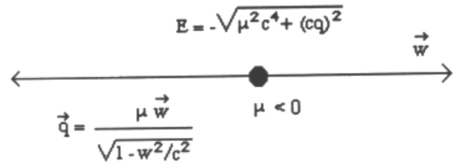

FIG. 10

The attributes of a negative energy boson.

As in Section II-1.2, we assume that attracting particles exhibit slight negative initial mass defects, δ_0 and $\delta'_0 < 0$ and that, once released at time t_0, they each emit a boson of fixed negative rest mass, μ and μ', recovering immediately their respective rest masses, m and m'.

III-1.3. The equations of motion.

In order to determine the dependence of Δ upon x, we proceed as in Section II-1. To this end we write the following recursive system of equations which is the counterpart of the set of equations [1] to [7] in case *I*. A double sign $\pm$ has been used to deal simultaneously with the repulsion (upper sign) or the attraction problem (lower sign).

Firstly the "energy momentum" subset (i = 0, 1, 2, ...):

$$[71a,b] \quad \begin{cases} \left[(m+\delta_i)^2 + p_{x,i}^2\right]^{1/2} = \left[m^2 + \pi_{x,i+1}^2\right]^{1/2} \pm (\mu_i^2 + q_{x,i+1}^2)^{1/2} \\ p_{x,i} = \pi_{x,i+1} + q_{x,i+1} \end{cases}$$

[72a,b] $$\begin{cases}\left[(m'+\delta'_i)^2+p_{x,i}^2\right]^{1/2}=\left[m'^2+\pi'^2_{x,i+1}\right]^{1/2}\pm(\mu'^2_i+q'^2_{x,i+1})^{1/2}\\ p_{x,i}=\pi'_{x,i+1}+q'_{x,i+1}\end{cases}$$

[73a,b] $$\begin{cases}\left(m^2+\pi^2_{x,i+1}\right)^{1/2}\pm(\mu'^2_i+q'^2_{x,i+1})^{1/2}=\left[(m+\delta_{i+1})^2+p_{x,i+1}^2\right]^{1/2}\\ p_{x,i+1}=\pi_{x,i+1}+q'_{x,i+1}\end{cases}$$

[74a,b] $$\begin{cases}\left(m'^2+\pi'^2_{x,i+1}\right)^{1/2}\pm(\mu^2_i+q^2_{x,i+1})^{1/2}=\left[(m'+\delta'_{i+1})^2+p_{x,i+1}^2\right]^{1/2}\\ p'_{x,i+1}=\pi_{x,i+1}+q_{x,i+1}\end{cases}$$

Because the two mediating bosons are successively and alternatively emitted and absorbed by both particles, we have used the following condensed notation,

[75a,b] $$\begin{cases}\mu_{2i}=\mu'_{2i+1}=\mu\\ \mu_{2i+1}=\mu'_{2i}=\mu'\end{cases}$$

Secondly the "space-time" subset is given by,

[76a,b] $$\begin{cases}x'_{i+1}-x'_i=v'_{x,i+1}(t'_{i+1}-t'_i)\\ x_{i+1}-x_i=v_{x,i+1}(t_{i+1}-t_i)\end{cases}$$

[77a,b] $$\begin{cases}x'_{i+1}=x_i+w_{x,i+1}(t'_{i+1}-t_i)\\ x_{i+1}=x'_i+w'_{x,i+1}(t_{i+1}-t'_i)\end{cases}$$

where the velocities of both particles are again given by equations [5a,b] and those of the bosons are related to the momenta in the following way,

[78a,b] $$\begin{cases}\vec{w}_{i+1}=\dfrac{\pm\vec{q}_{i+1}}{(\mu_i^2+q_{i+1}^2)^{1/2}}\\ \vec{w}'_{i+1}=\dfrac{\pm\vec{q}'_{i+1}}{(\mu'^2_i+q'^2_{i+1})^{1/2}}\end{cases}$$

The solution of equations [71] to [78] is unique provided the following initial conditions are known, the masses m and m', $t_0=t'_0=0$ without loss of generality and x_0 and x'_0. Though the choice of the origin is unimportant it may be judicious to associate it with the center of mass of the system correctly defined. The following quantities are also given, the initial momentum p_0, which is zero if the particles are initially at rest, the rest-masses, μ and μ', of the mediating bosons and the initial mass excesses δ_0 and δ'_0. To share the total mass deviation between its two components, δ_0 and δ'_0, we need again a conjecture similar to that of paragraph II-1.5.

III-1.2. The condition of simultaneity.

We postulate that the rest-masses of the bosons and the mass excesses initially borne by the particles, under the form δ_0 and δ'_0, are such that each two-step cycle, as described by Fig. 9, starts and ends simultaneously for both particles in the zero momentum frame.

That this simultaneity can hold is not at all evident. The demonstration needs again an in-depth discussion of equations [71] to [78]. The following is a shortened paraphrase of what has been done in Section II-1.2. Lengthy proofs are omitted for the sake of brevity.

Let us simply mention that the simultaneity condition implies again a relation between successive velocities, in fact the generalization of equation (21), which is written as:

$$\frac{v_{x,2i+1}v'_{x,2i+1} - w_{x,2i+1}w'_{x,2i+1}}{w_{x,2i+1} + w'_{x,2i+1} - v_{x,2i+1} - v'_{x,2i+1}} = \frac{v_{x,2i+2}v'_{x,2i+2} - w_{x,2i+2}w'_{x,2i+2}}{w_{x,2i+2} + w'_{x,2i+2} - v_{x,2i+2} - v'_{x,2i+2}}$$

Eliminating $p_{x,i+1}$ and the square roots between equations [66] to [73] leaves only four independant equations which generalizes equations [23] to [26]:

$$4m^2 q^2_{x,i+1} - 4(z_i - \mu_i^2)\pi_{x,i+1}q_{x,i+1} + 4\mu_i^2(\pi^2_{x,i+1} + m^2) - (z_i - \mu_i^2)^2 = 0$$

$$4m'^2 q'^2_{x,i+1} - 4(z'_i - \mu_i'^2)\pi'_{x,i+1}q'_{x,i+1} + 4\mu_i'^2(\pi'^2_{x,i+1} + m'^2) - (z'_i - \mu_i'^2)^2 = 0$$

$$4m^2 q'^2_{x,i+1} - 4(z_{i+1} - \mu_i'^2)\pi_{x,i+1}q'_{x,i+1} + 4\mu_i'^2(\pi^2_{x,i+1} + m^2) - (z_{i+1} - \mu_i'^2)^2 = 0$$

$$4m'^2 q^2_{x,i+1} - 4(z'_{i+1} - \mu_i^2)\pi'_{x,i+1}q_{x,i+1} + 4\mu_i^2(\pi'^2_{x,i+1} + m'^2) - (z'_{i+1} - \mu_i^2)^2 = 0$$

The definitions [8a, b] of the quantities z_i remain valid, i.e.,

$$z_i = 2m\delta_i + \delta_i^2 \quad \text{and} \quad z_i' = 2m'\delta_i' + \delta_i'^2$$

The compatibility condition for these equations, which generalizes equation [27], is written as,

$$[79] \qquad m'^2(z_i - \mu_i^2)(z_{i+1} - \mu_{i+1}^2) = m^2(z'_i - \mu_i'^2)(z'_{i+1} - \mu'^2_{i+1})$$

The theorems proven in Section II in the context of photon exchanges may be converted in the context of finite mass boson exchanges. They cover both the repulsive and the attractive cases with the appropriate signs of δ_i and δ'_i. Their proofs, which are based on arguments similar to those developed in Section II, are not reproduced here.

Theorem 8. The simultaneity condition, which ensures that a two-step cycle which starts at simultaneous times $t_{2i}=t'_{2i}$ for both masses will also end simultaneously, i.e.: $t_{2i+2}=t'_{2i+2}$, is automatically fulfilled if one respects two conditions, (1) the masses, μ and μ', of the bosons are inversely proportional to the masses of the particles,

[80] $$m\mu = m'\mu'$$

and (2) the initial mass excesses obey the generalized sharing law,

[81] $$\delta_0(2m+\delta_0)-\mu^2 = \delta'_0(2m'+\delta'_0)-\mu'^2$$

Theorem 9. Assuming that the condition [80] and [81] are satisfied, one verifies that if the relation,

[82] $$\delta_{2i}(2m+\delta_{2i})-\mu^2 = \delta'_{2i}(2m'+\delta'_{2i})-\mu'^2$$

holds for $i=0$, it holds for all successive i-values (1, 2, 3, ...).

The same can be said about the following equivalent expression,

$$(m+\delta_{2i})^4-2(m^2+\mu^2)(m+\delta_{2i})^2+(m^2-\mu^2)^2 =$$
$$(m'+\delta'_{2i})^4-2(m'^2+\mu'^2)(m'+\delta'_{2i})^2+(m'^2-\mu'^2)^2$$

Theorem 10. If the relation,

[83] $$x_{cm,2i}[(m+\delta_{2i})^2+p_{x,2i}^2]^{1/2} = -x'_{cm,2i}[(m'+\delta'_{2i})^2+p_{x,2i}^2]^{1/2}$$

holds for $i=0$, it holds for all successive i-values (1, 2, 3, ...).

Because equation [83] defines the center of mass of the system, it is equivalent to saying that this center of mass remains at rest or, in other words, that the ratio, $x'_{cm,2i}/x_{cm,2i} = x'_{cm,0}/x_{cm,0}$, remains constant.

Theorem 11. The following invariant quantities define the spatial dependence of the mass excesses,

[84a,b]
$$x_{cm,2i}[(m+\delta_{2i})^4-2(m^2+\mu^2)(m+\delta_{2i})^2+(m^2-\mu^2)^2]^{1/2} = C^t$$
$$x'_{cm,2i}[(m'+\delta'_{2i})^4-2(m'^2+\mu'^2)(m'+\delta'_{2i})^2+(m'^2-\mu'^2)^2]^{1/2} = C^t$$

Theorem 12. Equations [84a, b] may be rewritten in terms of the mutual distance between the particles:

[85] $$(x_{2i}-x'_{2i})[(m+\delta_{2i})^4-2(m^2+\mu^2)(m+\delta_{2i})^2+(m^2-\mu^2)^2]^{1/2} = C^t$$

All these results may be verified numerically using Mathematica's multiprecision arithmetics. A program is presented in the next section.

II-1.3. The numerical solution of the equations of motion.

Equations [71] and [72] are easily solved after elimination of $\pi_{x,i+1}$ and $\pi'_{x,i+1}$. One finds,

$$\begin{cases} q_{x,i+1} = \dfrac{p_{x,i}(z_i+\mu_i^2)\pm W_i\sqrt{(z_i-\mu_i^2)^2-4m^2\mu_i^2}}{2(m^2+z_i)} \\ q'_{x,i+1} = \dfrac{p'_{x,i}(z'_i+\mu_i^2)\mp W'_i\sqrt{(z'_i-\mu_i'^2)^2-4m'^2\mu_i'^2}}{2(m'^2+z'_i)} \end{cases}$$

The upper (lower) signs before the radicals correspond to the repulsive (attractive) case and have been choosen in order to entail the correct sign of the momenta q_x and q'_x. The complete solution results of a straigthforward resolution of the subsequent equations [73] to [78].

The total initial mass excess $\Delta_0=\delta_0+\delta'_0$ must be shared between the two particles in agreement with the relation (81). This leads to the following equivalent initialization,

$$[86] \quad \delta_0 = \frac{\Delta_0(\Delta_0+2m')+\mu^2-\mu'^2}{2(\Delta_0+m+m')} \text{ and } \delta'_0 = \frac{\Delta_0(\Delta_0+2m)-\mu^2+\mu'^2}{2(\Delta_0+m+m')}$$

where Δ_0 is given and it depends on the strength of the coupling at the starting point.

The following program computes recursively the numerical solutions of the repulsion problem with the following initial conditions, m=1; m'=2; μ=0.0005; $\mu'=\mu/2$=0.00025; p_0=0.02; δ_0=0.01.

```
mu[i_]:=mapr Mod[i,2]+ma Mod[i+1,2];mupr[i_]:=ma Mod[i,2]+mapr Mod[i+1,2];
m=1;mpr=2;ma=N[0.0005,25]; mapr=m ma/mpr;
d[0]=N[0.01,25];dpr[0]=N[-mpr+Sqrt[mpr^2+mapr^2-ma^2+d[0]^2+2m d[0]],25];
z[0]=d[0]^2+2m*d[0];zpr[0]=dpr[0]^2+2mpr*dpr[0];
px[0]=N[0.02,25];pprx[0]=-N[0.02,25];
En=N[Sqrt[(m+d[0])^2+px[0]^2]+Sqrt[(mpr+dpr[0])^2+px[0]^2],25];
W[0]=N[(En^2-mpr^2+m^2-mapr^2+ma^2)/(2En),25];
Wpr[0]=N[(En^2+mpr^2-m^2+mapr^2-ma^2)/(2En),25];
t[0]=0;tpr[0]=0;
x[0]=-1;xpr[0]=N[Sqrt[(m+d[0])^2+px[0]^2] x[0]/Sqrt[(mpr+dpr[0])^2
                                                    +px[0]^2],25];
xcm[0]=N[(x[0]-xpr[0])Wpr[0]/En,25];xprcm[0]=N[(xpr[0]-x[0])W[0]/En,25];

loop=Do[{{qx[i+1]=(px[i](z[i]+mu[i]^2)+W[i]*
Sqrt[(z[i]-mu[i]^2)^2-4m^2 mu[i]^2])/(2(m^2+z[i])),
qprx[i+1]=(pprx[i](zpr[i]+mupr[i]^2)-Wpr[i]*
```

```
Sqrt[(zpr[i]-mupr[i]^2)^2-4mpr^2 mupr[i]^2])/(2(mpr^2+zpr[i]))},
{pix[i+1]=px[i]-qx[i+1],piprx[i+1]=pprx[i]-qprx[i+1]},
{rad[i+1]=Sqrt[m^2+pix[i+1]^2],radpr[i+1]=Sqrt[mpr^2+piprx[i+1]^2]},
{px[i+1]=pix[i+1]+qprx[i+1],pprx[i+1]=piprx[i+1]+qx[i+1]},
{W[i+1]=rad[i+1]+Sqrt[mupr[i]^2+qprx[i+1]^2],
Wpr[i+1]=radpr[i+1]+Sqrt[mu[i]^2+qx[i+1]^2]},
{z[i+1]=W[i+1]^2-m^2-px[i+1]^2,zpr[i+1]=Wpr[i+1]^2-mpr^2-pprx[i+1]^2},
{d[i+1]=-m+(m^2+z[i+1])^(1/2),dpr[i+1]=-mpr+(mpr^2+zpr[i+1])^(1/2)},
{vx[i+1]=pix[i+1]/rad[i+1],vprx[i+1]=piprx[i+1]/radpr[i+1]},
{wx[i+1]=qx[i+1]/Sqrt[mu[i]^2+qx[i+1]^2],
wprx[i+1]=qprx[i+1]/Sqrt[mupr[i]^2+qprx[i+1]^2]},
{tpr[i+1]=(x[i]-xpr[i]+vprx[i+1]tpr[i]-wx[i+1]t[i])/(vprx[i+1]-wx[i+1]),
t[i+1]=(xpr[i]-x[i]+vx[i+1]t[i]-wprx[i+1]tpr[i])/(vx[i+1]-wprx[i+1])},
{xpr[i+1]=xpr[i]+vprx[i+1]*(tpr[i+1]-tpr[i]),x[i+1]=x[i]+vx[i+1]*(t[i+1]-t[i])},
If[Mod[i+1,2]==0,{xcm[i+1]=(x[i+1]-xpr[i+1])*
Wpr[i+1]/En,xprcm[i+1]=(xpr[i+1]-x[i+1])W[i+1]/En},a=0]}, {i,0,100}]
```

Fig.11 presents, in the exemplative case μ=0.0005, the evolution of $\Delta_{2i}=\delta_{2i}+\delta'_{2i}$ at the times t_{2i} as a function of the mutual distance: $r_{2i}=|x'_{2i}-x_{2i}|$.

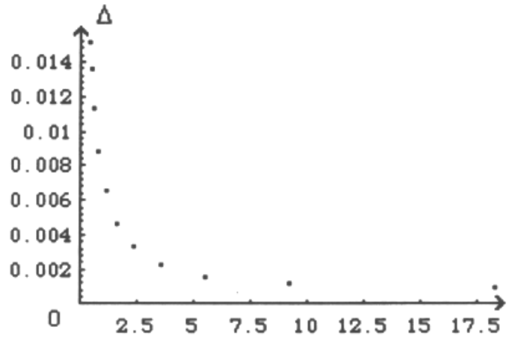

FIG. 11

The $1/r^2$ decrease of the short range repulsion potential in one dimension.

II-1.4. The analytical solution of the equations of motion.

Theorems 11 allows one to deduce the spatial dependence of δ_{2i} or δ'_{2i}. We recall that the total mass excess, $\Delta_{2i}=\delta_{2i}+\delta'_{2i}$ (i=0,1,2,...), is very close to the mutual potential energy, U_{2i}, of the system. In short, we find the generalization of equation [86],

$$[87] \quad \delta_{2i}=\frac{\Delta_{2i}(\Delta_{2i}+2m')+\mu^2-\mu'^2}{2(\Delta_{2i}+m+m')} \text{ and } \delta'_{2i}=\frac{\Delta_{2i}(\Delta_{2i}+2m)-\mu^2+\mu'^2}{2(\Delta_{2i}+m+m')}$$

Thanks to theorems 10 and 12, we deduce the spatial dependence of Δ_{2i} and we directly write the final result as a function of the mutual distance, $r_{2i}=|x'_{2i}-x_{2i}|$. This is the function, rewritten with usual units, which interpolates exactly the dots of Fig. 11:

[88]
$$\Delta/c^2 = -(m+m') + [m^2+\mu^2+(4m^2\mu^2+\lambda/r^2)^{1/2}]^{1/2}$$
$$+[m'^2+\mu'^2+(4m'^2\mu'^2+\lambda/r^2)^{1/2}]^{1/2}$$

In that general formula, $\lambda>0$ correspond to the repulsive case and $\lambda<0$ to the attractive case.

III-3 Extension of the model to higher dimensions: an application to the modified Kepler problem

It is interesting to extend the model to a two-dimensional interaction and to compare the results to those obtained in the context of the Kepler problem discussed in section II-3. For the sake of simplicity, we assume that the attracting center is infinitely heavy and therefore at rest. The simplification in that case is that the infinitely heavy center never emits nor absorbs a boson. Only the moving particle repeatidly emits and absorbs the mediating boson which is regularly simply reflected by the center. It seems unnecessary to reproduce Fig. 7 since the modifications are so slight. At time $t=t_0$, the particle experiences a negative mass defect, $\delta_0<0$, which is the mass equivalent of its attractive potential energy. Its momentum is $\vec{p}_0$ and its position, with respect to the attracting center, C, is $\vec{r}_0$. At time $t=t_0+\varepsilon$, the particle emits an antiboson of rest-mass μ in the direction of the center which reflects it so that it collides with the particle at the later time, $t=t_1$, at its new position, $\vec{r}_1$. It is then instantaneously reabsorbed so that the particle experiences a modified negative mass defect, $\delta_1<0$, and so on.

The evolution of the system is governed by the following sets of equations (i=0,1,2,...),

[89a,b]
$$\begin{cases} \left[(m+\delta_i)^2+p_i^2\right]^{1/2} = \left[m^2+\pi_{i+1}^2\right]^{1/2} - \left[\mu^2+q_{i+1}^2\right]^{1/2} \\ \vec{p}_i = \vec{\pi}_{i+1} + q_{i+1}\hat{r}_i \end{cases}$$

[90a,b]
$$\begin{cases} \left[m^2+\pi_{i+1}^2\right]^{1/2} - \left[\mu^2+q_{i+1}^2\right]^{1/2} = \left[(m+\delta_{i+1})^2+p_{i+1}^2\right]^{1/2} \\ \vec{p}_{i+1} = \vec{\pi}_{i+1} - q_{i+1}\hat{r}_{i+1} \end{cases}$$

[91a,b]
$$\begin{cases} \vec{r}_{i+1} = \vec{r}_i + \vec{\pi}_{i+1}(t_{i+1}-t_i)/\left[(m^2+\pi_{i+1}^2)\right]^{1/2} \\ r_i + r_{i+1} = q_{i+1}(t_{i+1}-t_i)/\left[\mu^2+q_{i+1}^2\right]^{1/2} \end{cases}$$

III-3.1. The solution of the modified Kepler problem.

To save space, we have not tried to solve this problem completely analytically. Rather, we content ourselves with a preliminary discussion which is sufficient to obtain the numerical solution in a recursive way. We use again the following notation,

$$\vec{r}_i = (r_i \cos\varphi_i, r_i \sin\varphi_i)$$

Two elementary constants of the motion exist once again, i.e. the total energy, W, and the angular momentum, J. They may be written in various equivalent ways,

$$[(m+\delta_i)^2 + p_i^2]^{1/2} = [m^2 + \pi_i^2]^{1/2} - [\mu^2 + q_i^2]^{1/2} = W \quad [92]$$

$$\vec{r}_i \times \vec{\pi}_i = \vec{r}_i \times \vec{\pi}_{i+1} = \vec{r}_i \times \vec{p}_i = \vec{J} \quad [93]$$

Squaring π_{i+1} in equation [89b] and combining with equation [92] leads to

$$2W[\mu^2 + q_{i+1}^2]^{1/2} = m^2 - \mu^2 - W^2 + p_i^2 - 2q_{i+1}\vec{p}_i \cdot \hat{r}_i$$

an equation which allows one to find q_{i+1}. Equation [89b] then furnishes π_{i+1}.

Calculations which are similar to those of Section II-3.1 lead to a relation which is equivalent to equation [62] in the photon exchange model,

$$\frac{1}{4}(r_i + r_{i+1})^2 \frac{m^2 q_{i+1}^2 - \mu^2 \pi_{i+1}^2}{q_{i+1}^2 (m^2 + \pi_{i+1}^2)} = r_i r_{i+1} - \frac{J^2(\mu^2 + q_{i+1}^2)}{m^2 q_{i+1}^2 - \mu^2 \pi_{i+1}^2}$$

a relation which may now be used to determine r_{i+1}. The value of $\vec{r}_{i+1}$ is then determined by equation [91] and we obtain $\vec{p}_{i+1}$ with the aid of equation [90b]. A new cycle, in which i is replaced by i+1 can then be started, achieving the recursive calculation of the main variables of the problem.

We have tested numerically the solution, starting with initial conditions which are similar to those of Section II-3.1, i.e., r_0=1, t_0=0, φ_0=0, p_0=0.02, δ_0=–0.001 and θ_0=angle($\vec{r}_0$,$\vec{p}_0$)=1.2radians. The rest mass of the mediating boson has been set to three distinct values: μ = –0.00005, –0.00010 and –0.00030, respectively. Special care must be taken to avoid numerical instability. The results of the computations are displayed in Fig. 12a,b,c. They show that the motion is quasiperiodic with a rotation of the perihelion of the ellipse which increases with μ. It also increases if the initial conditions are modified so that the particle enters into the critical domain near the origin.

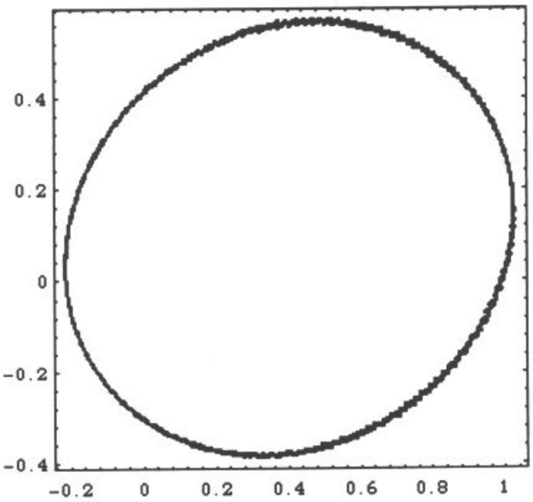

FIG. 12A

Three examples of modified Keplerian trajectories. (μ=–0.00005)

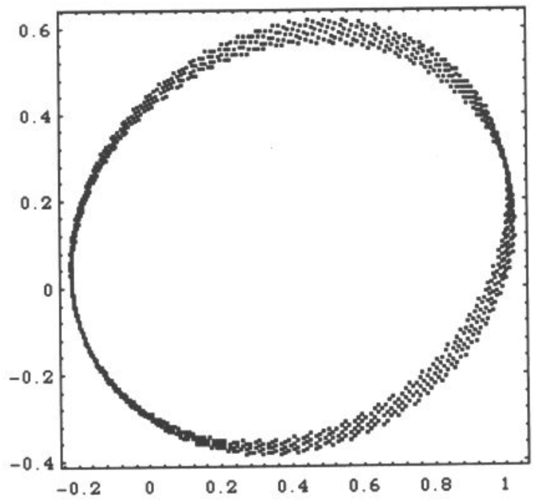

FIG. 12B

Three examples of modified Keplerian trajectories. (μ=–0.00010)

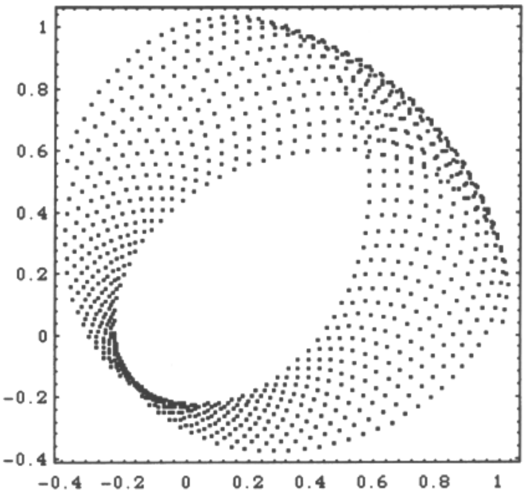

FIG. 12C

Three examples of modified Keplerian trajectories. (μ=–0.00030)

IV. Interpretation of the results and conclusions

We have shown that recursive dynamics, based on repeated boson exchange between interacting particles, effectively leads to well defined laws of motion. The obtained results coincide with or differ from the classical ones in various aspects, aspects which deserve a careful review. Herein we provide an extended discussion of various interesting aspects of the problem.

All our calculations are exact and relativistically invariant. This means that the trajectories that we obtained may be compared to those which are calculated in the frame of the theory of special relativity. For example, considering the Kepler problem with a fixed center, we have retreived the classical precessing ellipse in the case of photon exchange. Moreover we have verified that the rotation of the perihelion corresponds to the expected value, with the only condition being that the particle does not enter into the critical region defined by equation [68].

To our knowledge, no exact solution of the same problem is known if the center is itself movable. This results because the magnetic effects are automatically superimposed on the Coulomb interaction, a superposition which makes the equations much more difficult to solve. In our model we have found, in that case, that the simultaneity condition becomes velocitydependent. The problem is however numerically solvable and we do not expect severe discrepancies compared with the results of customary special relativity in the non-critical domain.

We have not considered the three body problem. Of course a solution can only be obtained numerically because the general problem is known to be chaotic. In our model the origin of that chaotic behaviour can be viewed as the consequence of the non-commutativity between successive collisions between the particles and the mediating bosons. Indeed, each particle receives bosons emitted by the two others. A slight alteration of the initial conditions may have the consequence that, at some time later, the order of two successive collisions is modified such that the particles pursue their own trajectory with different momentum and energy. This sensitivity to initial conditions is one of the ingredients of chaos. Special cases already known by Lagrange are however exactly solvable. Although we have not verified it, we expect that a somewhat relaxed simultaneity condition may be restored in these special, highly symmetric, configurations.

We observe that, in our model, the exact calculations are purely algebraic, contrarily to what happens in relativistic mechanics in which the equations of motion require proper integration. A non-relativistic version of this model could perfectly well have been discovered by Newton in order to explain accurately the motion of the planets. We will never know how physics would have evolved in such a case.

An essential characterisitics of our model is the absence of distinction between the interaction law and the law of motion. In classical physics one writes first an evolution equation, usually in the form $\vec{F}=d\vec{p}/dt$, and then one incorporates the force law. Our model is simply based on the conservation of the relativistic energy-momentum four-vector without any assumption concerning the shape of an eventual force law. In fact no force law is necessary because there is no force at all! Similarly, the acceleration does not play any crucial role in the model.

In the same way that no force field exists in our model, there is no need for a potential law. The entire model is sustained by the existence of mass deviations carried by the interacting particles, either in excess or in defect, according to the type of interaction, repulsive or attractive. The distant interaction is then understood as the consequence of repeated exchanges of mediating bosons. No fundamental distinction is made in this respect between the electric and the gravitational interactions. This opens interesting views about a possible unification of both types of interactions. For example the magnetic effects which are easily observable in electrodynamics would find an automatic equivalent in the gravitational two-body problem. Easy estimates however indicate that it would be a very slight correction in usual astronomical problems.

Because interacting particles experience very short uniform rectilinear motion between successive collisions, it becomes evident that the solutions which are obtained in the frame of our model must differ, in some way, from the classical solutions. It is only in the limiting case where c, the speed of light, would tend to infinity, that we should retreive the usual differential equations of motion. This results because the time recursion step would tend to zero. The expected discrepancy is however small unless the particles enter the critical domains defined by equation [68]. We may estimate the difference by observing that the total mass deviation, $\Delta_{2i} = \delta_{2i} + \delta'_{2i}$, is equivalent to the usual potential energy, U_{2i}, far from the origin. This means that the asymptotic behaviours of Δ and U are identical. In case I, for photon exchange, we have effectively retrieved the $1/r$ asymptotic decrease. However, in case II, for finite mass boson exchange, we have found a short-range potential law [88] which exhibits an unexpected $1/r^2$ asymptotic behaviour. This result is a little surprising because it is often thought that finite rest mass boson exchange should necessarily lead to a Yukawa-type(10) potential, $U(r)=A\exp(-\lambda r)/r$. On contrast, our calculations are void of approximation so that we have no apparent reason to doubt the $1/r^2$ asymptotic law. The complete analysis of the situation near the origin is much more delicate because the approximation, ΔU, is no longer valid. Let us consider in greater detail the example of the photon exchange. Equations [32] and [56] reveal that the usual potential energy, U_{2i}, is equal to $\nu_{2i} + \nu'_{2i}$, in one dimension, and to ν_i in the fixed center Kepler problem. The exact calculation of these quantities in terms of the distance between the interacting particles is possible in both cases.

For example, we have found respectively,

$$U_{2i} = \nu_{2i} + \nu'_{2i} = C^{t} \frac{2mm'\sqrt{r_{2i-2}r_{2i}} + (m^2 + m'^2)r_{2i}}{mm'\sqrt{r_{2i-2}r_{2i}} - \lambda} \frac{m}{r_{2i}}$$

and,

$$U_i = \nu_i = \frac{1}{1 + (m^2/H)(r_i + r_{i-1})}$$

It should be noted that the retarded effects, r_{2i-2}, are absent in the corresponding formulas for the total mass deviation Δ_{2i}. A first interesting point is that the laws that are found for U are not universal, in contrast to what happens with Δ. This clearly means that, in this model, the mass deviation rather than the potential energy is the essential quantity. If one persist in being interested in U, another essential point is its r-dependence near the origin which avoids the usual $1/r$ singularity. That modified potential law has no consequence in the study of solar gravitation because the critical distance has been found to be very short in this case. The only restriction might occur shortly after the bigbang, if one agrees that all the universe was concentrated in a very small space. This paper will not discuss the possible cosmological implications of the modified behaviour of U near the origin. We simply mention that the discussion is open to an eventual modification of Newton's laws in the context of the unsolved problem of the missing mass in the universe. The modified law also deserves attention in the domain of electrodynamics. In particular, it can reasonably be expected that the $2S_{1/2}$-$2P_{1/2}$ degeneracy observed in the resolution of Dirac's equation should be broken, leading to a theoretical estimate of the Lamb shift in the hydrogen atom. This degeneracy is indeed characteristic of the $1/r$ potential law. That the $1/r$ singularity of U near the origin has been removed might also be of interest in all the domains where it is thought to be troubleshooting.

Let us however mention that, at very short distances, we leave the classical domain. A discrepancy may arise in the quantum domain when the distance approaches the de Broglie length of the interacting particles so that a complete understanding of what really occurs very near the origin can be reached only when the variable number of mediating bosons is taken into account(11).

We have systematically neglected the eventual radiation effects when the particles accelerates in an inertial frame.

As a final remark, we would like to emphasize that we have implicitely assumed that the total energy of an interacting particles may be written as,

[94] $$W = [(m+\delta)^2c^4+c^2p^2]^{1/2}.$$

In other words, we have considered that the mass deviation possesses the inertial property so that we must include it in the rest mass of the interacting particle before calculating its kinetic energy. This point of view may be justified a posteriori by the neatness of the results that we have obtained. It may also be justified in a more heuristical way as follows. Everybody agrees that the inertial rest mass, m', of a system of interacting particles, say two, is lower than the sum, m, of the inertial masses of its constituents. Examples are the hydrogen atom or the deuteron. When such a system moves at a velocity, v, its relativistic total energy is classically calculated with equation [94] on the basis of the modified inertial mass, $m'=m+\delta$. Now, if we consider one interacting particle inside the system, it seems reasonable to calculate its total energy, in the same way, i.e., according to formula [94].

In conclusion, recursive dynamics considers that interacting particles carry positive or negative mass deviations. In order to try to recover their original rest-mass, they regularly exchange bosons and evolves therefore in space according to the relativistic conservation laws of energy and momentum. The methodology is applicable to all kind of interactions so that a door is open for a new way of considering their unification.

Appendix A.

We prove, in the context of the Kepler problem studied in section II-3, that the following quantity is invariant for all i = 0, 1, 2, ... ,

$$[A1]\qquad 2W\left[r_i r_{i+1}\right]^{1/2}\cos\left[(\varphi_{i+1}-\varphi_i)/2\right]-m(r_{i+1}+r_i)=H/m$$

Proof: we first eliminate $\vec{p}_i$ between [53b] and [54b] and then substitute ν_i given by equ. [53a],

$$[A2]\ \vec{\pi}_{i+1}-\vec{\pi}_i=(\nu_i+\nu_{i+1})\hat{r}_i=\left(-\left[m^2+\pi_{i+1}^2\right]^{1/2}-\left[m^2+\pi_i^2\right]^{1/2}+2W\right)\hat{r}_i$$

On the other side, eliminating (t_{i+1}-t_i) between equ. [55a] and [55b] funishes

$$\vec{\pi}_{i+1}/\left[m^2+\pi_{i+1}^2\right]^{1/2}=(\vec{r}_{i+1}-\vec{r}_i)/(r_{i+1}+r_i)$$

Squaring that equation allows one to extract π_{i+1}^2 so that one finds

$$[A3]\qquad \vec{\pi}_{i+1}=(m/2)(\vec{r}_{i+1}-\vec{r}_i)/\left[(r_i r_{i+1})^{1/2}\cos\left[(\varphi_{i+1}-\varphi_i)/2\right]\right]$$

That relation is useful to calculate successively,

$$[A4]\ \vec{\pi}_{i+1}+\left[m^2+\pi_{i+1}^2\right]^{1/2}\hat{r}_i=(m/2)(\vec{r}_{i+1}+r_{i+1}\hat{r}_i)/\left[(r_i r_{i+1})^{1/2}\cos\left[(\varphi_{i+1}-\varphi_i)/2\right]\right]$$

$$[A5]\ \vec{\pi}_i-\left[m^2+\pi_i^2\right]^{1/2}\hat{r}_i=-(m/2)(\vec{r}_{i-1}+r_{i-1}\hat{r}_i)/\left[(r_{i-1}r_i)^{1/2}\cos\left[(\varphi_i-\varphi_{i-1})/2\right]\right]$$

Eliminating $\vec{\pi}_1$ and $\vec{\pi}_{i+1}$ between equ. [A2], [A4] and [A5] and scalar multiplying the result by $\hat{r}_1$ leads to

$$[A6]\ 2W\sqrt{r_i}=m\sqrt{r_{i-1}}\cos\left[(\varphi_i-\varphi_{i-1})/2\right]+m\sqrt{r_{i+1}}\cos\left[(\varphi_{i+1}-\varphi_i)/2\right]$$

Multiplying that equation by the sign-conjugate of its right-hand side and rearranging the order of the terms gives

$$[A7]\quad \begin{aligned}&2W(r_{i-1}r_i)^{1/2}\cos\left[(\varphi_i-\varphi_{i-1})/2\right]-mr_{i-1}\cos^2\left[(\varphi_i-\varphi_{i-1})/2\right]=\\&2W(r_i r_{i+1})^{1/2}\cos\left[(\varphi_{i+1}-\varphi_i)/2\right]-mr_{i+1}\cos^2\left[(\varphi_{i+1}-\varphi_i)/2\right]\end{aligned}$$

On the other side, is not too difficult to prove that the following relation holds, as a consequence of equ. [60]:

$$[A8]\ \ r_i+r_{i-1}-r_i-r_{i+1}=r_{i-1}\cos^2\left[(\varphi_i-\varphi_{i-1})/2\right]-r_{i+1}\cos^2\left[(\varphi_{i+1}-\varphi_i)/2\right]$$

Eliminating the $\cos^2$-terms between [A7] and [A8] leads to the invariant expression which achieves the proof.

Appendix B.

Here we prove theorem 7. We start by squaring equ. [A4]:

[B1] $$\left(\vec{\pi}_{i+1}+\left[m^2+\pi_{i+1}^2\right]^{1/2}\hat{r}_i\right)^2=m^2 r_{i+1}/r_i$$

On the other side, combining equ. [53a] and [53b] leads to

[B2] $$\vec{p}_i=\vec{\pi}_{i+1}+\left[m^2+\pi_{i+1}^2\right]^{1/2}\hat{r}_i-W\hat{r}_i$$

Comparing [B1] and [B2], we get

[B3] $$(\vec{p}_i+W\hat{r}_i)^2=p_i^2+W^2+2W\hat{r}_i\cdot\vec{p}_i=m^2 r_{i+1}/r_i$$

or equivalently, on account of [53b] and [A4],

[B4] $$\left[r_i r_{i+1}-J^2/m^2\right]^{1/2}=(r_i W+\vec{r}_i\cdot\vec{p}_i)/m$$

We modify progressively the invariant [63], using successively [B4], [B3] and finally [53b]:

$$-H=m^2(r_{i+1}+r_i)-2Wm\left[r_i r_{i+1}-J^2/m^2\right]^{1/2}=m^2 r_{i+1}+m^2 r_i-2W(r_i W+\vec{r}_i\cdot\vec{p}_i)$$
$$=r_i p_i^2+r_i W^2+2W\vec{p}_i\cdot\vec{r}_i+m^2 r_i-2Wr_i W-2W\vec{r}_i\cdot\vec{p}_i=r_i(m^2+p_i^2-W^2)$$

The final result may be rewritten as:

$$r_i(2m\delta_i+\delta_i^2)\;=\;H$$

a result which proves theorem 7.

Appendix C.

The proof of theorem 9 results because the z_i and the z'_i obey to the first order recurrence,

$$(m/m')[(z'_{i+1}-\mu'^2_{i+1})/(z_{i+1}-\mu^2_{i+1})]=1/\{(m/m')[(z'_i-\mu'^2_i)/(z_i-\mu_i^2)]\}$$

Its solution is immediate as,

[C1] $$\begin{cases} z_{2i}-\mu^2=z'_{2i}-\mu'^2 \\ m'^2(z_{2i+1}-\mu'^2)=m^2(z'_{2i+1}-\mu^2) \end{cases}$$

Hence one has,

$$\delta_{2i}(2m+\delta_{2i})-\mu^2=\delta'_{2i}(2m'+\delta'_{2i})-\mu'^2$$

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