Classical Monopoles: Newton, NUT-space, gravomagnetic lensing and atomic spectra

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Abstract

Stimulated by a scholium in Newton's Principia we find some beautiful results in classical mechanics which can be interpreted in terms of the orbits in the field of a mass endowed with a gravomagnetic monopole. All the orbits lie on cones! When the cones are slit open and flattened the orbits are exactly the ellipses and hyperbolae that one would have obtained without the gravomagnetic monopole.

The beauty and simplicity of these results has led us to explore the similar problems in Atomic Physics when the nuclei have an added Dirac magnetic monopole. These problems have been explored by others and we sketch the derivations and give details of the predicted spectrum of monopolar hydrogen.

Finally we return to gravomagnetic monopoles in general relativity. We explain why NUT space has a non-spherical metric although NUT space itself is the spherical space-time of a mass with a gravomagnetic monopole. We demonstrate that all geodesics in NUT space lie on cones and use this result to study the gravitational lensing by bodies with gravomagnetic monopoles.

We remark that just as electromagnetism would have to be extended beyond Maxwell's equations to allow for magnetic monopoles and their currents so general relativity would have to be extended to allow torsion for

general distributions of gravomagnetic monopoles and their currents. Of course if monopoles were never discovered then it would be a triumph for both Maxwellian Electromagnetism and General Relativity as they stand!

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I. INTRODUCTION

One of us was asked to review Chandrasekhar's (1995) book on Newton's Principia (1687) for Notes and Records of the Royal Society (Lynden-Bell 1996). This led to reading passages of Cajori's translation of Principia. In his first proposition Newton shows that motion under the influence of a central force will be in a plane and that equal areas will be swept by the radius vector in equal times. In his second proposition he shows that if a radius from a point S to a body sweeps out equal areas in equal times then the force is central. There follows this scholium: "A body may be urged by a centripetal force compounded of several forces; in which case the meaning of the proposition is that the force which results out of all tends to the point S. But if any force acts continually in the direction of lines perpendicular to the *described surface*, this force will make the body to deviate from the plane of its motion; but it will neither augment nor diminish the area of the *described surface* and is therefore to be neglected in the composition of forces."

What does this mean?

The words *described surface* have been translated from a Latin word that carries the extra connotation of a surface described by its edge. We shall take this to be the surface swept out by the radius vector to the body that is now describing the non-coplanar path. A force normal to this surface at the body must be perpendicular to $\mathbf{r}$ and $\mathbf{v}$ which are both within the surface, so Newton is considering extra forces of the form $Nm_0\mathbf{r} \times \mathbf{v}$ where N may depend on $\mathbf{r}, \mathbf{v}, t$ etc. We write the equation of motion

$$m_0 d^2\mathbf{r}/dt^2 = -V'(r)\hat{\mathbf{r}} + N\,\mathbf{L}, \tag{1.1}$$

where $V(r)$ is the potential for the central force, $\hat{\mathbf{r}}$ is the unit radial vector, and

$$\mathbf{L} = m_0\mathbf{r} \times \mathbf{v}\,. \tag{1.2}$$

Taking the cross product $\hat{\mathbf{r}} \times (1.1)$ we have

$$d\mathbf{L}/dt = N\mathbf{r} \times \mathbf{L} \tag{1.3}$$

from which it follows either geometrically à la Newton or by dotting with $\mathbf{L}$ that

$$|\mathbf{L}| = \text{ constant.} \tag{1.4}$$

Now if φ is the angle *measured within the described surface* between a fixed half-line ending at S and the radius vector,

$$\tfrac{1}{2}r^2\dot{\varphi} = \tfrac{1}{2}|\mathbf{L}|/m_0 \tag{1.5}$$

so equal areas are swept out in equal times just as Newton says. To see this angle more precisely it is perhaps worthwhile to work in axes which are continually tilting to keep up with the plane of the motion. In any axes rotating with angular velocity $\mathbf{\Omega}(t)$, the apparent acceleration $\ddot{\mathbf{r}}$ is related to the absolute acceleration $d^2\mathbf{r}/dt^2$ by

$$d^2\mathbf{r}/dt^2 = \ddot{\mathbf{r}} + 2\mathbf{\Omega} \times \dot{\mathbf{r}} + \dot{\mathbf{\Omega}} \times \mathbf{r} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})\,. \tag{1.6}$$

We shall apply this formula to axes which are always tilting about $\hat{\mathbf{r}}$ such that in these axes the motion appears as planar. Thus putting $\mathbf{\Omega} = \Omega\hat{\mathbf{r}}$ in Eq. (1.6)

$$d^2\mathbf{r}/dt^2 = \ddot{\mathbf{r}} + \Omega r^{-1}\mathbf{L}/m_0\,. \tag{1.7}$$

Inserting this into Eq. (1.1) and choosing

$$\Omega = rN, \tag{1.8}$$

we recover in these axes the equation we would have had in inertial axes had Newton's extra force $\propto N$ been absent i.e.,

$$m_0\ddot{\mathbf{r}} = -V'\hat{\mathbf{r}}\,. \tag{1.9}$$

Thus relative to *these moving axes* $\mathbf{r} \times m_0\dot{\mathbf{r}} = \mathbf{L}$ is constant not only in magnitude but also in direction and

$$|\mathbf{r} \times \dot{\mathbf{r}}| = r^2\dot{\varphi} = L/m_0 \tag{1.10}$$

where φ is the angle at S between some line fixed in the moving axes and the current radial line (this is of course equal to the earlier angle since this moving plane is 'rolling' on the *described surface* about the common radius vector).

We now return to the inertial axes in which the direction of $\mathbf{L}$ varies in accord with (1.3). Dotting Eq. (1.1) with $\mathbf{v} = d\mathbf{r}/dt$ the N term goes out so the energy equation is left unchanged and we have, remembering that L^2 is constant,

$$\frac{m_0}{2}\mathbf{v}^2 + V = \frac{m_0}{2}\left[\dot{r}^2 + \left(\frac{L}{m_0}\right)^2 r^{-2}\right] + V = E\,. \tag{1.11}$$

Here $\dot{r}$ is the same in fixed or rotating axes since this r is scalar. Equation (1.11) demonstrates that the radial motion $r(t)$ is precisely that which would have occurred had N been zero. Furthermore (1.9) and (1.10) demonstrate that within the tilting axes, or [using (1.5)] within the described surface, the solution $r(\varphi)$ is precisely the same function that we would have found for the truly planar motion that occurs with N absent. Although this extension of Newton's theorem is not in Principia it would surprise us if Newton had not seen and understood it. There is interesting historical research to be done here on Newton's surviving manuscripts. We know from Whiteside that this scholium was not in the first draft of Newton's De Motu Corporum written in Autumn 1684 but appears in its revision which is probably dated to the Spring of 1685.

Although (1.5) and (1.11) are sufficient for the solution of the motion within the described surface, we need to find that surface by solving (1.3) for a complete description of the motion. This is not particularly simple and to do it we need to prescribe how N depends on $\mathbf{r}, \mathbf{v}, t$ etc. However $d\mathbf{L}/dt$ and $d\hat{\mathbf{r}}/dt$ are always parallel since both are perpendicular to $\mathbf{r}$ and $\mathbf{L}$. This led us to consider under what circumstances they might be proportional. In particular

$$d\hat{\mathbf{r}}/dt = -\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{v})/r = -\mathbf{r} \times \mathbf{L}/(r^3 m_0) \tag{1.12}$$

so in full generality we have from (1.3)

$$d\mathbf{L}/dt = -(m_0 N r^3) d\hat{\mathbf{r}}/dt\,. \tag{1.13}$$

This demonstrates that when $m_0 N r^3 = Q_* =$ constant we have a beautifully simple solution to (1.13) to wit.

$$\mathbf{L} + Q_*\hat{\mathbf{r}} = \mathbf{j} = \text{const} \tag{1.14}$$

here $\mathbf{j}$ is the vector constant of integration; notice that Q_* has the same dimensions as L. Since $\mathbf{L}$ and $\hat{\mathbf{r}}$ are perpendicular we deduce, dotting with $\hat{\mathbf{r}}$

$$\mathbf{j} \cdot \hat{\mathbf{r}} = Q_* \tag{1.15}$$

which shows that the angle between $\mathbf{j}$ and $\hat{\mathbf{r}}$ is constant so $\hat{\mathbf{r}}$ moves on a cone whose axis is along $\mathbf{j}$. Similarly dotting (1.14) with $\mathbf{L}$ we find $L^2 = \mathbf{j} \cdot \mathbf{L}$ so likewise $\mathbf{L}$ moves on another cone with $\mathbf{j}$ as its axis. If this cone has semi-angle χ then $L/|\mathbf{j}| = \cos\chi$, but $\hat{\mathbf{r}}$ and $\mathbf{L}$ are orthogonal and by (1.14) they are coplanar with $\mathbf{j}$ so we may choose cf. (1.15)

$$Q_*/|\mathbf{j}| = \sin\chi \tag{1.16}$$

so the angle between $\hat{\mathbf{r}}$ and $\mathbf{j}$ is $\pi/2 - \chi$ as shown in Fig. 1. Notice from (1.16) that the angle of the cone is determined completely from $|\mathbf{L}|$ and the force constant Q_*. Orbits with larger $|\mathbf{j}|$ have smaller χ so the angular momentum then moves around a narrow cone and $\hat{\mathbf{r}}$ then moves around a very open one. For $|\mathbf{j}| \gg Q_*$ it is almost planar. Fig. 1 illustrates two circular orbits moving in opposite senses about the same axis. Notice that the one moving right-handedly about the upward pointing axis is displaced above the center sitting like a halo about it while that moving left handedly is displaced below the center like an Elizabethan ruff below the head. One might have supposed that for $j \gg Q_*$ these two circular orbits would approach the central plane but although the cone becomes much flatter and more open the displacement between the direct and retrograde orbits actually increases. For circular orbits at distance a from S we have, for a Newtonian potential, $L^2 = GMam_0^2$ and the displacement is

$$2\hat{\mathbf{j}} \cdot \mathbf{r} = a/\sqrt{GMam_0^2 Q_*^{-2} + 1} \to m_0^{-1} Q_* \sqrt{a/(GM)}.$$

We have been led to the case $m_0 N r^3 = Q_* =$ constant for reasons of mathematical simplicity but this case is more than a mathematical curiosity because:

1. Of all the forces of Newton's N type [see Eq. (1.1)] only those of the form $-\mathbf{v} \times \hat{\mathbf{r}} r^{-2} Q_*(\theta, \phi)$ derive from a Lagrangian. For a monopole Q_* is constant.

2. We may rewrite this force in the form

$$N\mathbf{L} = -Q_* \mathbf{v} \times \mathbf{r}/r^3 = \frac{m_0}{c} \mathbf{v} \times \mathbf{B}_g, \tag{1.17}$$

where

$$\mathbf{B}_g = -Q\hat{\mathbf{r}}/r^2 \quad ; \quad Q = Q_* c/m_0. \tag{1.18}$$

We have introduced the velocity of light c to make the analogy with magnetic forces even more obvious. $\mathbf{B}_g$ is clearly the field of a magnetic monopole of strength Q but since this sort of magnetism acts not on moving charges but rather on moving masses; it is a gravomagnetic field. Such fields are well known in general relativity see Landau & Lifshitz Theory of Fields (1966) §89 problem 1. They are position dependent Coriolis forces associated with what relativists less helpfully call the dragging of inertial frames. The field $\mathbf{B}_g$ as we have defined it has the same dimensions as $\mathbf{g}$ the acceleration due to gravity and Q/G has the dimensions of mass. In electricity, like charges repel while in gravity, like masses attract. It is the same with like magnetic monopoles, they repel while the gravomagnetic monopoles of like sign attract one another, hence the negative sign in Eq. (1.18) is best left there rather than combined into a new definition of the pole strength Q. We may find the Lagrangian corresponding to the force (1.17) by analogy with the electrodynamic case. There we add a term $q\mathbf{v} \cdot \mathbf{A}/c$ where q is the charge and $\mathbf{A}$ is the vector potential. For any poloidal axi-symmetric magnetic field one may choose $\mathbf{A}$ to be of the form $A\nabla\phi$ where ϕ is the azimuth around the axis. We require

$$-Q\hat{\mathbf{r}}/r^2 = \mathbf{B}_g = \nabla \times (A\nabla\phi) = \nabla A \times \nabla\phi \tag{1.19}$$

from which one readily finds $A = Q(1 + \cos\theta)$ gives the right $\mathbf{B}_g$. Thus a Lagrangian for Eq. (1.1) is

$$\mathcal{L} = \tfrac{1}{2} m_0 v^2 - m_0 V(r) + Q_*(1 + \cos\theta)\mathbf{v} \cdot \nabla\phi. \tag{1.20}$$

Although the dynamical system is spherically symmetrical the Lagrangian is not and can not be made so. The only spherically symmetrical vector fields are $f(r)\mathbf{r}$. If $\mathbf{A}$ were of this form its curl would be zero and therefore could not be the field of a monopole. Of course we can choose any axis we like and measure θ and ϕ appropriately from it. The $\mathbf{A}$ field will then be quite different but it will give the same $\mathbf{B}_g$ field by construction. Thus the difference between any two such $\mathbf{A}$ fields will have zero curl showing that $\mathbf{A}' = \mathbf{A} + \nabla\chi$ i.e., a gauge transformation. The Lagrangian (1.20) is neither spherically symmetrical nor gauge invariant but it is a member of a whole class of equivalent Lagrangians with different axes which are related by gauge transformations. Whereas none of these is individually spherical the class of all of them is spherically symmetric. The moral is that it can be restrictive to impose symmetry on a single member of the class if the member is not gauge invariant.

So far everything holds for any spherical potential $V(r)$. We could for example choose it to be Henon's (1959) isochrone potential $2aV_0/(a+\sqrt{r^2+a^2})$ or its better known limits the simple harmonic oscillator $a \gg r$ or the Newtonian potential $a \ll r$. For all isochrones the orbits can be solved using only trigonometric functions (see e.g., Lynden-Bell 1963, Evans *et al.* 1990). Here we shall stick to the Newtonian potential $V/m_0 = -GM/r$. We have already shown that the motion lies on a cone whose semi-angle is given by $\cos^{-1}(Q_*/|\mathbf{j}|)$; furthermore if we slit that cone along $\varphi = 0$ and flatten it, the orbit will be exactly what it would have been in the absence of N i.e., a conic section. Of course when we slit and flatten the orbit's cone a gap appears whose angle is $\gamma = 2\pi\left[1 - L/\sqrt{L^2+Q_*^2}\right]$, see Fig. 2. An ellipse with focus at S and apocentre at $\varphi = 0$ would get back to apocentre at $\varphi = 2\pi$ but unfortunately the gap intervenes. On the cone we identify $\varphi = 0$ not with $\varphi = 2\pi$ but rather with $\varphi = 2\pi - \gamma$. Thus on the cone the ellipse will precess forwards by an angle γ in each radial period, Fig. 3. This angle γ is an angle like φ measured at S within the cone's surface. It is perhaps more natural to measure angles η around the axis of the cone; these angles are related through $\dot{\eta} = \dot{\varphi}/\cos\chi = L/(m_0 r^2 \cos\chi)$ so $\eta = \varphi \sec\chi = \varphi|\mathbf{j}|/L$.

In these terms the precession per radial period is

$$\Delta\eta = 2\pi(|\mathbf{j}|/L - 1). \tag{1.21}$$

Newton in his proposition on revolving orbits showed that the addition of an inverse cube force led to an orbit of exactly the same shape but traced relative to axes that rotate at a rate proportional to $\dot{\phi}$ in the original orbit. It is natural to ask whether such an additional force can stop the precession around the cone of an orbit in the monopolar problem and so yield an orbit that closes on itself in fixed axes. Wonderfully a simple change in $V(r)$ does this not just for one orbit but for all orbits at once. We thus obtain a new superintegrable system in which all bound orbits close. By analogy with Hamilton's derivation of his eccentricity vector (Hamilton 1847) we take the cross product of the equation of motion (1.1) with $\mathbf{j} = \mathbf{L} + Q_* \hat{\mathbf{r}}$. On the right hand side two terms are zero and the remaining two are multiples of $d\hat{\mathbf{r}}/dt$ cf. (1.12) so we find

$$m_0 \mathbf{j} \times d^2\mathbf{r}/dt^2 = -(m_0 r^2 V' + Q_*^2 r^{-1}) d\hat{\mathbf{r}}/dt. \tag{1.22}$$

This will integrate vectorially if the bracket is constant. Calling it GMm_0^2 we find the potential must be of the form

$$V/m_0 = -GMr^{-1} + \frac{1}{2}\frac{Q_*^2}{m_0^2} r^{-2}. \tag{1.23}$$

Evidently the required inverse cube repulsive force is proportional to the square of the monopole moment Q. Integrating (1.22) we have

$$d\mathbf{r}/dt \times \mathbf{j} = GMm_0(\hat{\mathbf{r}} + \mathbf{e}) \tag{1.24}$$

where $\mathbf{e}$ is the vector constant of integration. Dotting (1.24) with $\hat{\mathbf{r}}$ we have

$$\ell_*/r = (\mathbf{l} + \mathbf{e} \cdot \hat{\mathbf{r}}) \tag{1.25}$$

where $\ell_* = \mathbf{L} \cdot \mathbf{j}/(GMm_0^2) =$ const. Equation (1.25) is the equation of a conic section of eccentricity $\mathbf{e}$ which defines the direction to pericentre. But we have not yet proved that the orbit lies in a plane so (1.25) actually defines a prolate spheroid, paraboloid or hyperboloid. Nick Manton, by analogy with his work on monopoles in Euclidean Taub Space (Gibbons & Manton 1986), showed us that the motion is in fact planar; for using (1.15) Eq. (1.25) becomes on multiplication by $Q_* r$

$$Q_* \ell_* = (\mathbf{j} + Q_* \mathbf{e}) \cdot \mathbf{r}$$

which demonstrates that the orbit lies on a plane whose normal is $\mathbf{j} + Q_* \mathbf{e}$. As $\mathbf{r}$ also lies on a cone this provides another proof that the motion lies along a conic section.

Notice that the vector integral $\mathbf{e}$ in (1.24) together with the integral $\mathbf{j}$ appears to provide six integrals of the motion. However they are not all independent because $-\mathbf{e} \cdot \mathbf{j} = \hat{\mathbf{r}} \cdot \mathbf{j} = Q_*$. So they provide 5 independent integrals. Thus we have a new superintegrable dynamical system in which the bound orbits exactly close (cf. Evans 1990, 91).

It was the beauty and simplicity of these results for monopoles in classical mechanics that led us to believe that a similar simplicity might well be discernible both in quantum mechanics and in general relativity. We were not disappointed, both had already attracted attention. Hautot 1972 discusses the separation of variables in r, θ, ϕ coordinates. The vector integral $\mathbf{j}$ is preferable because the generality of motion on cones is then seen. For motion in special relativity $\mathbf{j}$ is still conserved provided $\mathbf{L}$ is interpreted as $m_0 \mathbf{r} \times d\mathbf{r}/d\tau = m_0 \mathbf{r} \times \mathbf{v}/\sqrt{1 - v^2/c^2}$. Goddard and Olive (1978) in their excellent review of monopoles in gauge field theories quote Poincaré (1895) for this integral in the classical case of a pure electromagnetic monopole.

II. DIRAC'S MONOPOLE AND THE SPECTRA OF MONOPOLAR ATOMS

A. Gauge transformations, Schrödinger's equation & Dirac's quantised monopole

The Lagrangian for a particle of mass m_0 and charge $-e$ in an electromagnetic field is

$$\mathcal{L} = \tfrac{1}{2} m_0 \dot{\mathbf{r}}^2 - e \dot{\mathbf{r}} \cdot \mathbf{A}/c + e \Phi(r)$$

where Φ is the electrostatic potential and $\mathbf{B} =$ Curl $\mathbf{A}$ is the magnetic field. The momentum conjugate to $\mathbf{r}$ is

$$\mathbf{p} = \partial \mathcal{L}/\partial \dot{r} = m_0 \dot{\mathbf{r}} - e\mathbf{A}/c$$

which is not a gauge invariant quantity. However the particle's momentum $m_0 \dot{\mathbf{r}} = \mathbf{p} + e\mathbf{A}/c$ is gauge invariant and therefore has greater physical significance. The Hamiltonian is given by

$$H = \mathbf{p}\cdot\mathbf{v} - \mathcal{L} = (2m_o)^{-1}(\mathbf{p} + e\mathbf{A}/c)^2 - e\Phi. \tag{2.1}$$

For Schrödinger's equation we replace $\mathbf{p}$ by $-i\hbar\nabla$ and solve $H\psi = E\psi$ for the wave function of a steady state. A given magnetic field $\mathbf{B}$ can be described by many different vector potentials $\mathbf{A}$ related by gauge transformations $\mathbf{A}' = \mathbf{A} + \nabla\chi$. Each will give us a different Hamiltonian. Let us first see how the different wave functions corresponding to these are related. Define a new function ψ' such that

$$\psi = \exp[ie\chi/(\hbar c)]\psi' \tag{2.2}$$

then $(-i\hbar\nabla + e\mathbf{A}/c)\psi = \exp[ie\chi/(\hbar c)](-i\hbar\nabla + e\mathbf{A}/c + e\nabla\chi/c)\psi'$ and the combination $\mathbf{A}' = \mathbf{A} + \nabla\chi$ has appeared. Applying the above operator twice we see that

$$H\psi = \exp[ie\chi/(\hbar c)]H'\psi'$$

where H' is H with $\mathbf{A}$ replaced by $\mathbf{A}'$. It follows that Schrödinger's equation $H\psi = E\psi$ implies $H'\psi' = E\psi'$ so under gauge transformation ψ transforms to ψ' given by (2.2).

This is a perfectly good wave function whenever χ is single valued but following Aharonov and Böhm (1959) we now consider the wave function of a particle outside a small impenetrable cylinder $R = a$. If we take $\nabla\chi = \nabla(F\phi/2\pi), R \geq a$ where ϕ is the azimuth this corresponds to the same magnetic field outside the cylinder but a different magnetic flux within it because

$$\int \mathbf{B}'\cdot d\mathbf{S} = \oint \mathbf{A}'\cdot d\boldsymbol{\ell} = \oint(\mathbf{A} + \nabla\chi)\cdot d\boldsymbol{\ell} = \int \mathbf{B}\cdot d\mathbf{S} + F$$

which identifies the constant F as the extra flux threading the cylinder. If we adopt our transformation of wave function for a gauge transformation we get the phase factor

$$\exp[-ieF\phi/(hc)] \tag{2.3}$$

which is only single valued when the flux takes the special values

$$F = N(hc/e) \tag{2.4}$$

where N is an integer (positive, negative or zero). (This N is not the force coefficient of Section I). Thus while we get the correct wave function for those particular values of F, we need to solve the problem anew with the correct boundary condition that ψ' must be periodic in ϕ whenever F is not an integer multiple of the flux quantum hc/e. Indeed when it is not, there is interference between the two parts of a beam of electrons that pass on either side of such a cylinder just because their phases differ by $e \oint \Delta\mathbf{A} \cdot d\boldsymbol{\ell}/(hc) = eF/(hc)$. It was just this phase shift that was observed in the experiments demonstrating the Aharanov Böhm effect of the magnetic flux even when the electron beams were untouched by the magnetic field. There is an intimate connection of this result with Dirac's (1931) earlier quantum of magnetic monopole from which one flux unit (2.4) emanates.. This comes about because in the presence of a monopole $\oint \mathbf{A} \cdot d\boldsymbol{\ell}$ is itself multivalued.

Consider the integral $\oint \mathbf{A} \cdot d\boldsymbol{\ell}$ around a small loop; this is clearly the flux of $\mathbf{B}$ through the loop but such a flux is ambiguous in the presence of a monopole since it depends on whether the surface spanning the loop is chosen to pass above or below the monopole i.e., S_1 or S_2 in Fig. 4. The difference between these two estimates is just $\int_{S_1 - S_2} \mathbf{B} \cdot d\mathbf{S} = 4\pi Q$ by Gauss's theorem. Inserting this $\Delta \int \mathbf{A} \cdot d\boldsymbol{\ell}$ in place of F in (2.3) we see that the wave function will only have an unambiguous phase provided

$$4\pi Q = N(hc/e) \tag{2.5}$$

i.e., provided that the monopole strength is quantized in Dirac units of $\frac{1}{2}\hbar c/e \approx \frac{137e}{2}$.

The quantum of magnetic flux (2.4) is inversely proportional to the charge. Quanta of half this size are observed in the Josephson effect in superconductivity where the effect is due to paired electrons. There is some evidence for the larger unit (2.4) in ordinary conductors at low temperatures (Umbach *et al.* 1986).

Returning to Schrödinger's equation (2.1) and using the vector potential [cf. under (1.19)]

$$\mathbf{A} = -Q(1 + \cos\theta)\nabla\phi \tag{2.6}$$

we have the correct magnetic field for a monopole of strength Q but we notice that $\mathbf{A}$ is singular along the line $\theta = 0$ although it is regular along $\theta = \pi$. Near the singular line $\mathbf{A} \to -2Q\nabla\phi$ which is the vector potential of a tube carrying a flux $4\pi Q$ downwards, thus formula (2.6) represents the vector potential of a magnetic monopole fed its flux by the singular half line $\theta = 0$. This half line gives an unobservable Aharanov-Böhm effect provided $4\pi Q = Nhc/e$ that is provided the monopole is a multiple of the Dirac (1931, 48) unit. Extra interest in his monopole comes from his argument that it can also be read as a reason for charge quantization, because, if Q is the least monopole, then e must be a multiple of $\hbar c/Q$; thus in his picture, charge quantization and monopole quantization spring from the same source. It is of interest that $\mathbf{A}$ in (2.6) is single valued. It has avoided the multi-valuedness alluded to above by having the singular string at $\theta = 0$ down to the monopole. This plays the role of the cut in multivalued functions in the complex plane Wat & Yang (1976). An interesting historical remark is that Schrödinger in 1922 saw that quantum conditions in the old quantum theory led to $\Gamma \equiv \frac{e}{c}[\oint \Phi dt - \mathbf{A} \cdot d\mathbf{x}] = nh$ while Weyl's gauge theory led him to consider $\exp\,(-\Gamma/\gamma)$ with γ as yet unspecified. He realized that the identification $\gamma = -i\hbar$ would lead naturally to such quantum numbers and after de Bröglie (1925), he built on this idea to invent his wave mechanics in 1926. (See Yang 1987).

B. Solution of Schrödinger's Equation

Written in spherical polar coordinates Schrödinger's equation is

$$-\frac{\hbar^2}{2m_0 r^2}\left\{\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\psi}{\partial\mu}\right] + \right.$$
$$\left. + \frac{1}{1-\mu^2}\left[\frac{\partial^2\psi}{\partial\phi^2} - iN(\mu+1)\frac{\partial\psi}{\partial\phi} - \tfrac{1}{4}N^2(\mu+1)^2\psi\right]\right\} -$$
$$-e\,\Phi\,\psi = E\psi \tag{2.7}$$

here μ has been written for $\cos\theta$ and N is the number of Dirac monopoles on the nucleus. ϕ only occurs as $\partial/\partial\phi$ in the above equation so we may take one Fourier component with $\psi \propto \exp\,(im\phi)$ and m an integer positive, negative or zero in order that ψ be single valued.

On multiplication by $-2m_0 r^2/(\hbar^2\psi)$ we then find the separated equation

$$\frac{1}{\psi}\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\psi}{\partial\mu}\right]-\frac{[m-N\frac{1}{2}(\mu+1)]^2}{1-\mu^2}=-C=$$
$$=-\frac{1}{\psi}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)-\frac{r^2 2m_0}{\hbar^2}(E+e\Phi). \tag{2.8}$$

Writing $\psi=\psi_r(r)\psi_\mu(\mu)$ the left hand side is a function of μ alone and the right hand side is a function of r alone so both must be a constant which we call $-C$. The resultant equation for ψ_μ has regular singular points at $\mu=\pm 1$. The indicial equations for the series solutions about $\mu=\pm 1$ have regular solutions behaving as $(1-\mu)^{\frac{1}{2}|m-N|}$ and $(1+\mu)^{\frac{1}{2}|m|}$ respectively, so we remove those factors by writing

$$\psi_\mu=(1-\mu)^{\frac{1}{2}|m-N|}(1+\mu)^{\frac{1}{2}|m|}F(\mu). \tag{2.9}$$

After some algebra the equation for F takes the form

$$(1-\mu^2)F''+[(|m|+1)(1-\mu)-(|m-N|+1)(1+\mu)]F'+$$
$$+\tfrac{1}{2}[2C-m(m-N)-|m||m-N|-|m-N|-|m|]F=0. \tag{2.10}$$

We now write $z=\frac{1}{2}(1+\mu)$, so $z(1-z)=(1-\mu^2)/4$ and $dz=\frac{1}{2}d\mu$ which reduces the above equation into the standard form for the hypergeometric equation i.e.,

$$z(1-z)d^2F/dz^2+[c-(a+b+1)z]\,dF/dz-abF=0 \tag{2.11}$$

where

$$c=|m|+1 \tag{2.12}$$

$$a+b=|m|+|m-N|+1 \tag{2.13}$$

and $-2ab$ is the final square bracket in Eq. (2.10).

The hypergeometric function finite at $\mu=-1, z=0$ diverges like $(1-\mu)^{c-a-b}$ at $\mu=1$, that is twice as fast as the first factor in (2.9) converges, so in order to get convergence the hypergeometric series must terminate. This occurs only if a or b is a negative integer or zero. w.l.g. taking $b = -K$ we find that F reduces to a Jacobi polynomial $P_K^{\alpha\beta}(\mu)$ so that ψ_μ takes the form

$$\psi_\mu = C_{kmn}(1-\mu)^{1/2|m-N|}(1+\mu)^{1/2|m|}P_K^{|m-N|,|m|} \tag{2.14}$$

here $\int_{-1}^{+1}\psi_\mu^2 d\mu = 1$ and C_{kmn} is the normalization

$$\left[\frac{(2K+|m-N|+|m|+1)K!(K+|m-N|+|m|)!}{2^{|m-N|+|m|+1}(K+|m-N|)!(K+|m|)!}\right]^{\frac{1}{2}}.$$

The condition that $b = -K$ gives

$$-2ab = 2K(|m|+|m-N|+1+K)$$

and hence (noticing that $K = 0$ leaves (2.14) finite) we have

$$C = K(K+1) + K(|m|+|m-N|)+$$
$$+\tfrac{1}{2}\left[m(m-N)+|m||m-N|+|m-N|+|m|\right]. \tag{2.15}$$

If we write $j = K + \frac{1}{2}(|m|+|m-N|)$, then we notice that j is a positive half-integer and $j \geq \frac{1}{2}(|m|+|m-N|)$

$$C = j(j+1) - N^2/4. \tag{2.16}$$

Thus C and j are only integers when N is an *even* integer. When N is odd C and j are an integer $\pm\frac{1}{2}$. For given j and $N \geq 0$, $m - \frac{N}{2}$ takes the $2j+1$ values from $-j$ to $+j$ in steps of 1. For $N = 1$ the ground state has $j = \frac{1}{2}$ and $C = \frac{1}{2}$ rather than the values 0 familiar from the normal hydrogen atom. The $j = \frac{1}{2}$ states with $m = 1$ and $m = 0$ are degenerate, see Fig. 5.

With the value (2.16) for C we now turn to the radial equation for ψ_r, (2.8). Here the treatment is very close to the classical case clearly laid out by Pauling and Wilson (1935). We take $\Phi = Ze/r$, Ze being the nuclear charge, and E negative. We write

$$\alpha^2 = -2m_0E/\hbar^2 \tag{2.17}$$

$$\zeta = m_0 Ze^2 h^{-2}\alpha^{-1} \tag{2.18}$$

and use a normalized radius $\tilde{r} = 2\alpha r$. As all the radii in the rest of this section are so normalized we shall forget the $\tilde{}$ and take it as read. Eq. (2.8) now takes the form

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi_r}{dr}\right) + \left(-\frac{1}{4} - \frac{C}{r^2} + \frac{\zeta}{r}\right)\psi_r = 0. \tag{2.19}$$

The asymptotic form of this equation for large r is $\psi_r'' = \psi_r/4$ so $\psi_r \to \exp(r/2)$ or $\exp -(r/2)$. Of these only the second is acceptable so we write

$$\psi_r = \exp(-r/2) r^s f(r) \tag{2.20}$$

where $f(r)$ may be expanded in series about the origin in the form

$$\sum_{p=0}^{\infty} a_p\, r^p$$

and s is chosen so that $a_0 \neq 0$. The indicial equation found by substitution of the series (2.20) into (2.19) is

$$[s(s+1) - C]a_0 = 0$$

but by hypothesis $a_0 \neq 0$ so using (2.16) s is given by

$$\left(s + \tfrac{1}{2}\right)^2 = \tfrac{1}{4} + s\,(s+1) = \tfrac{1}{4} + C = \left(j + \tfrac{1}{2}\right)^2 - \tfrac{1}{4}N^2. \tag{2.21}$$

The recurrence relation for general p is then

$$p(p + 2s + 1)a_p = (s + p - \zeta)a_{p-1}$$

and the asymptotic form for large p is $a_p \to a_{p-1}/p$ which shows that $f \to e^r$. In that case ψ_r would diverge at large r. This is unacceptable so the series must terminate. Thus there must be a positive integer $p = n' + 1$ such that

$$\zeta = p + s = n' + s + 1 \tag{2.22}$$

with s given by (2.21). Returning to (2.18) and (2.17) this gives the eigenvalues for the energy in the form

$$E = -\frac{m_0 Z^2 e^4}{2\hbar^2}\frac{1}{(n' + s + 1)^2} = -\frac{m_0 Z^2 e^4}{2\hbar^2}\frac{1}{(n + \Delta)^2} \tag{2.23}$$

where $n = n' + J + 1$ where J takes values 0, 1, 2 ... replaces the usual $\boldsymbol{\ell}$ and

$$J = j - \tfrac{1}{2}|N| \geq 0$$

N is the number of Dirac monopoles on the nucleus and

$$\Delta = \sqrt{(J+\tfrac{1}{2})(J+\tfrac{1}{2}+|N|)} - \left(J+\tfrac{1}{2}\right) \quad \geq 0.$$

Notice that Δ depends on J as well as $|N|$ and is only zero when $N = 0$. For large $J/|N|$,

$$\Delta \to \tfrac{1}{2}|N| \left[1 - \frac{1}{4}\frac{|N|}{J+\frac{1}{2}} + \frac{1}{8}\left(\frac{|N|}{J+\frac{1}{2}}\right)^2 - \cdots\right],$$

while for the ground state $J = 0$

$$\Delta = \tfrac{1}{2}\left(\sqrt{2|N|+1} - 1\right)$$

which becomes $\frac{1}{2}\left(\sqrt{3}-1\right)$ for $N = 1$. So Δ is *not* small. For a spinless electron the degeneracy of a state of given J and n is $2j+1 = 2J+1+|N|$ with $m - \frac{1}{2}|N|$ taking values from $-j$ to $+j$. Notice that the ground state $J = 0, n = 1$ is a doublet for $N = 1$ and has j value $\frac{1}{2}$ with $m = 0$ and $m = +1$ states even *before* we have allowed for further degeneracy due to electron spin. A single Dirac monopole thus gives some effects reminiscent of spin $\frac{1}{2}$ particles (Goldhaber 1976).

The dependence of Δ upon J lifts the degeneracy of the different J states (ℓ states) that occurs in normal hydrogen. The energy levels are near to those for an atom with a true spinning electron laid out in Figs. 6, 7, 8 and Tables 1 & 2. The degeneracy would return if the extra repulsive potential $\frac{1}{2}Q^2/(m_0 r^2 c^2)$ were included. Then the $-\frac{1}{4}N^2$ in (2.21) would be cancelled so s would become equal to j.

C. Selection Rules

The string to the monopole makes it look non-spherical but this is not truly the case as putting the string in any other direction can be achieved by a mere gauge transformation.

Therefore without loss of generality we may evaluate transition moments by taking the displacement in the z direction in which case we get

$$R_{ab} = \int \psi_a^* r\mu\psi_b d^3r =$$
$$= 2\pi\delta_{m_a m_b} \int_0^\infty \psi_{ra}\psi_{rb} r^3 dr \int_{-1}^{+1} \psi_{\mu a}\mu\psi_{\mu b} d\mu.$$

The radial integral is that for normal hydrogen but the scales have changed since s in (2.22) is no longer ℓ but is given instead by (2.21). We shall concentrate on the important change in selection rules given by the final integral.

Whereas for the Legendre Polynomials in normal hydrogen wave functions we have

$$\mu P_\ell(\mu) = \frac{\ell+1}{2\ell+1} P_{\ell+1} + \frac{\ell}{2\ell+1} P_{\ell-1}$$

so that $\int_{-1}^{+1} P_{\ell'}\mu P_\ell d\mu$ is only non-zero when $\ell' - \ell = \pm 1$, for the Jacobi polynomials in monopolar hydrogen

$$\mu P_K^{\alpha\beta} = \left(a_1 P_{K+1}^{\alpha\beta} + a_2 P_K^{\alpha\beta} + a_4 P_{K-1}^{\alpha\beta} \right) / a_3$$

where

$$a_1 = 2(K+1)(K+\alpha+\beta+1)(2K+\alpha+\beta)$$
$$a_2 = (2K+\alpha+\beta+1)(\alpha^2-\beta^2)$$
$$a_3 = (2K+\alpha+\beta)(2K+\alpha+\beta+1)(2K+\alpha+\beta+2)$$
$$a_4 = 2(K+\alpha)(K+\beta)(2K+\alpha+\beta+2)$$

so that $\int_{-1}^{+1}(1-\mu)^\alpha(1+\mu)^\beta P_{K'}^{\alpha\beta}\mu P_K^{\alpha\beta} d\mu$ will be non-zero when $K' - K = \pm 1 \,\text{or}\, 0$. [The 0 term is only absent when $a_2 = 0$ i.e., $\alpha \equiv |m - N| = \beta \equiv |m|$. This occurs for $N = 0$ always, for $N = 2$ when $m = 1$, but never for $N = 1$.]

Thus there is a significant change in the selection rules for electric dipole transitions. Some might imagine that magnetic dipole transitions should be important but the magnetic monopole is on a heavy nucleus and barely responds to an oscillating magnetic field so it is still the electric dipole transitions of the electron that are important. m is unchanged for a dipole along the z-axis so $\Delta K = \pm 1$ or 0 leads directly to Δj and hence $\Delta J = \pm 1$ or 0 for such transitions.

Even order of magnitude estimates show that the interaction of the electron spin's magnetic moment with unit monopole gives not a delicate fine structure but significant changes in the eigenvalues! Thus to find the true eigenvalues the Dirac equation is a necessity! Before treating it we clear up some details. We took the form (2.2) for $\mathbf{A}$ corresponding to a monopole with a string along $\mu = +1$. For $|N| \geq 2$ we could have taken two or more inwardly directed strings of flux. Are these different string configurations really different monopoles or do they all give the same eigenvalues? The effect of such a change is to add a unit flux string along the z axis. It is simple to show that this is equivalent to adding one to m everywhere that it occurs. Provided we do that also to m in the definition of j under (2.15) the final spectrum remains unchanged. What does change are the K and m values associated with a given j value.

A second detail is the value of m_0 which for $N = 0$ would be the reduced mass of the electron so for hydrogen it is $m_0 = m_e m_p/(m_e + m_p)$.

Particle physicists expect a heavy mass for any monopole so any monopolar hydrogen will have a nucleus much heavier than the proton and m_e should be substituted for m_0 in predicting spectra. A third detail for later reference is the energy spectrum of the relativistic Klein-Gordon equation. Here we follow Schiff's treatment and obtain writing $\alpha_z = Ze^2/(\hbar c)$

$$E = m_0 c^2 \left\{ \left[1 + \frac{\alpha_z^2}{(n + \Delta_1)^2} \right]^{-1/2} - 1 \right\} \tag{2.25}$$

where

$$\Delta_1 = \sqrt{(J + \tfrac{1}{2})(J + \tfrac{1}{2} + |N|) - \alpha_z^2} - \left(J + \tfrac{1}{2}\right). \tag{2.26}$$

D. Angular Momentum

Returning to the classical conserved quantity cf. (1.14) we see the conserved quantity is not the particle's angular momentum $\mathbf{L} = \mathbf{r} \times m_0 \mathbf{v}$ but rather that supplemented by $eQc^{-1}\hat{\mathbf{r}}$. The physics behind this supplement lies in the Poynting vector of the electromagnetic field which carries an angular momentum

$$\frac{1}{4\pi c}\int \mathbf{r}'\times\left(\mathbf{E}\times\frac{Q}{r'^2}\hat{\mathbf{r}}'\right)d^3r' = \frac{Q}{4\pi c}\int(\mathbf{E}\cdot\nabla)\,\hat{\mathbf{r}}'d^3r' =$$
$$= \frac{-Q}{4\pi c}\int \hat{\mathbf{r}}'\nabla\cdot\mathbf{E}d^3r' = +\frac{eQ}{c}\hat{\mathbf{r}} = +\tfrac{1}{2}N\hbar\hat{\mathbf{r}}$$

where $\nabla\cdot\mathbf{E} = -e4\pi\delta^3(\mathbf{r}'-\mathbf{r})$. The total angular momentum is thus $\mathbf{j} = \mathbf{L} + eQc^{-1}\hat{\mathbf{r}}$.

As we saw above (2.1) $m_0\mathbf{v} = \mathbf{p} + e\mathbf{A}/c$ in the presence of a magnetic field so the operator representing $\mathbf{j}$ is $\mathbf{r}\times(-i\hbar\nabla + e\mathbf{A}/c) + \frac{1}{2}N\hbar\hat{\mathbf{r}}$. The commutators $[-i\hbar\partial_j + eA_j/c, -i\hbar\partial_k + eA_k/c] = -i\hbar e\,c^{-1}\varepsilon_{jkl}B_l = -i\hbar eQc^{-1}\varepsilon_{jkl}x^l/r^3$ and $[-i\hbar\partial_j + eA_j/c, x^k] = -i\hbar\partial_j^k$ enable one to derive the commutator

$$[j_j, j_k] = i\hbar\varepsilon_{jkl}j_l$$

which demonstrates that $\mathbf{j}$ obeys the angular-momentum algebra of the rotation group. One may also demonstrate that $\mathbf{j}^2$ commutes with $\mathbf{j}$ and that $j_\pm = j_x \pm ij_y$ are the raising and the lowering operators for j_z. From which it follows by the usual argument that the eigenvalues of j_z are $-j\hbar$ to $+j\hbar$ and that the eigenvalues of $\mathbf{j}^2$ are $j(j+1)\hbar^2$. But $|\mathbf{j}|^2 = |\mathbf{L}|^2 + \frac{1}{4}N^2\hbar^2$ so the eigenvalues of $|\mathbf{L}|^2$ are $\left[j(j+1) - \frac{1}{4}N^2\right]\hbar^2$. Now looking at our separation of variables expression (2.8) we see that the LHS is just $-\hbar^{-2}|\mathbf{L}^2|$ by construction and hence $C = \left(j(j+1) - \frac{1}{4}N^2\right)$ which agrees with (2.16) and identifies the j defined there with the generalized angular momentum eigenvalue defined in this section. Note that for a single Dirac monopole and a non-spinning electron we showed (2.16) that j took *half odd integer values.*

In the next section we look at the Dirac equation for a spinning electron. There the correct generalization is $\mathbf{j} = \mathbf{L} + \frac{1}{2}N\hbar\hat{\mathbf{r}} + \frac{1}{2}\hbar\boldsymbol{\sigma}$.

This new $\mathbf{j}$ obeys the angular momentum algebra of the rotation group but now its eigenvalues are $j(j+1)\hbar^2$ with j taking integer (or half integer) values $\geq \frac{N+1}{2}$ depending on whether N is odd or even.

The Dirac equation may be written in standard notation

$$H\psi = \left[-c\boldsymbol{\alpha}\cdot(\mathbf{p}+e\mathbf{A}/c) - \beta m_0 c^2 + V\right]\psi = E\psi$$

with the newly defined $\mathbf{j}$ we find $\frac{d\mathbf{j}}{dt} = [\mathbf{j}, H] = 0$ so that each component of this generalized $\mathbf{j}$ commutes with the Hamiltonian always provided that $\mathbf{A}$ is a vector potential for the monopole. Following Schiff's treatment (1955) we define $p_r = r^{-1}(\mathbf{r}\cdot\mathbf{p} - i\hbar)$ and $\alpha r = r^{-1}(\boldsymbol{\alpha}\cdot\mathbf{r})$ and $\hbar\mathbf{k} = \beta\sigma'\cdot\left(\mathbf{r}\times\left(\mathbf{p}+\frac{e\mathbf{A}}{c}\right)+\hbar\right)$. No $\mathbf{A}$ term is needed in p_r since $\mathbf{r}\cdot\mathbf{A} = 0$ for our monopole.

The Hamiltonian is now written

$$H = -c\alpha_r p_r - i\frac{\hbar c}{r}\alpha_r\beta k - \beta m_0 c^2 + V$$

and as before α_r, β and p_r all commute with $\mathbf{k}$. The eigenvalues of $\mathbf{k}$ follow by squaring the definition

$$\hbar^2 k^2 = (\boldsymbol{\sigma}'\cdot\mathbf{L})^2 + 2\hbar\boldsymbol{\sigma}'\cdot\mathbf{L} + h^2 = L^2 + \tfrac{1}{4}\hbar^2.$$

In the last section we showed that L^2 has eigenvalues $j(j+1)\hbar^2 - \frac{N^2}{4}\hbar^2$ where j was an integer (N odd) or half odd integer (N even) so k^2 has eigenvalues $\left(j+\frac{1}{2}\right)^2 - \frac{1}{4}N^2$. Save for this change of k the usual separation of the Dirac equation goes through unscathed and following Schiff one obtains the energy levels

$$E = m_0 c^2\left\{\left[1+\frac{\alpha_z^2}{(s+n')^2}\right]^{-1/2} - 1\right\}$$

where $s = (k^2-\alpha_z^2)^{1/2}$ and $\alpha_z = Ze^2/(\hbar c)$.

n' is the radial quantum number. Inserting our eigenvalues $k^2 = \left(j+\frac{1}{2}\right)^2 - \frac{1}{4}N^2$ with $j = J + \frac{1}{2}(|N|+1)$ and $J = 0, 1, 2,$ etc. we have

$$E = m_0 c^2\left\{\left[1+\frac{\alpha_z^2}{(n+\Delta)^2}\right]^{-1/2} - 1\right\}$$

where $n = n' + J + 1$ and

$$\Delta = \sqrt{(J+1)(J+1+|N|) - \alpha_z^2} - (J+1).$$

These energy levels were first derived by Hautot (1972, 73) generalizing Harish-Chandra's (1968) separation of the Dirac equation[1]. For the scattering by monopoles see also Goldhaber (1965), Kazama & Yang (1976) and Kazama *et al.* (1977). We have drawn the bound energy levels that result Figs. 6, 7 and 8 and derived the wavelengths of the lines of "Monopolar Hydrogen" with one or two Dirac monopoles attached to the nucleus. Tables 1 & 2. Schwinger (1966) has suggested that the unit monopole should have the strength of two Dirac monopoles. With colleagues he has also calculated the motions of charged monopoles, dyons, under their mutual attraction (Schwinger *et al.* 1976). While monopoles may seem esoteric it is worthwhile looking for lines of monopolar hydrogen in the spectra of exotic astronomical objects.

III. GRAVOMAGNETIC MONOPOLES IN GENERAL RELATIVITY, NUT SPACE

A. NUT space the general spherically symmetric gravity field

Zelmanov (1956) and Landau & Lifshitz (1966) in developing their very physical approach to general relativity consider stationary space-times and put the metric in the form

$$ds^2 = e^{-2\nu}(dx^0 - A_\alpha dx^\alpha)^2 - \gamma_{\alpha,\beta} dx^\alpha dx^\beta \tag{3.1}$$

where $\nu \geq 0, A_\alpha$ and $\gamma_{\alpha\beta}$ are independent of $x^0 = ct$. (Our ν is $-\frac{1}{2}\nu$ of Landau & Lifshitz).

However this form is not unique since a transformation of time zero $x'^0 = x^0 + \chi(x^\alpha)$ leads to

$$ds^2 = e^{-2\nu}(dx'^0 - A'_\alpha dx^\alpha)^2 - \gamma_{\alpha\beta} dx^\alpha dx^\beta$$

where $A'_\alpha = A_\alpha + \nabla_\alpha \chi$ so under such a change $\mathbf{A}$ undergoes a gauge transformation. Landau & Lifshitz also show that $\gamma_{\alpha\beta}$ can be regarded as a metric of space i.e., the quotient space

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[1]For the further generalization to the problem with an additional Böhm-Aharonov string, see Villalba (1994, 95) & Hoaïg *et al.* (1992).

V^4/L^1 (Geroch 1970) – as opposed to space-time. They show that test bodies following geodesics of space-time depart from the geodesics of space as if acted on by gravitational forces which in our notation take the form

$$\mathbf{f} = \frac{m_0}{\sqrt{1 - v^2/c^2}} \left[\mathbf{E}_g + \frac{\mathbf{v}}{c} \times e^{-\nu} \mathbf{B}_g \right] \tag{3.2}$$

where the gravitational field

$$\mathbf{E}_g = c^2 \nabla \nu \tag{3.3}$$

and

$$\mathbf{B}_g = c^2 \text{ Curl } \mathbf{A}. \tag{3.4}$$

The conserved energy of the particle in motion is

$$\varepsilon = m_0 c^2 e^{-\nu} \left(1 - v^2/c^2\right)^{-1/2} \tag{3.5}$$

$e^{-\nu} < 1$ is the redshift factor by which energy is degraded.

Rewriting Landau & Lifshitz's form of Einstein's equations (§95 problem) we find

$$\text{div } \mathbf{B}_g = 0 \tag{3.6}$$

$$\text{Curl } \mathbf{E}_g = 0 \tag{3.7}$$

$$\text{div } \mathbf{E}_g = -c^{-2} \left[4\pi G \frac{(\rho c^2 + 3p) + \frac{v^2}{c^2}(\rho c^2 - p)}{1 - v^2/c^2} - \right.$$
$$\left. -\tfrac{1}{2} e^{-2\nu} \mathbf{B}_g^2 - \mathbf{E}_g^2 \right] \tag{3.8}$$

where ρ is the energy density in the rest frame of the fluid, $3p$ is the trace of its pressure tensor and v its velocity defined locally by local time synchronized along the fluid's motion. For non-relativistic velocities this equation reduces to Poisson's equation with the primary term on the right being $4\pi G\rho$. The remaining term has the form of a negative energy density contributed by the gravity fields. The next equation takes the form

$$\text{Curl } (e^{-\nu} \mathbf{B}_g) = -c^{-3} [16\pi G \mathbf{j}_g - 2c \mathbf{E}_g \times e^{-\nu} \mathbf{B}_g]. \tag{3.9}$$

Notice that $e^{-\nu}\mathbf{B}_g$ occurs also in that combination in the expression for the force. It is attractive to regard the final term as an energy current corresponding to a Poynting vector flux of gravitational field energy. $\mathbf{j}_g$ the matter energy current is given by

$$\mathbf{j}_g = \frac{\rho c^2 + p}{1 - v^2/c^2}\ \mathbf{v}.$$

The final Landau & Lifshitz equation for the 3 stress tensor is

$$P^{\alpha\beta} - E_g^{\alpha;\beta} = \left(T^{\alpha\beta} + \tfrac{1}{2}\dot{T}\gamma^{\alpha\beta}\right) + \\ + e^{-2\nu}(B_g^\alpha B_g^\beta - B_g^2\gamma^{\alpha\beta}) + E_g^\alpha E_g^\beta, \tag{3.10}$$

$P^{\alpha\beta}$ is the 3 dimensional Ricci Tensor constructed from the metric $\gamma^{\alpha\beta}$. Those familiar with the Maxwell stresses of magnetic and electric fields in say magnetohydrodynamics will find some interest in the field terms on the right. The matter terms may be rewritten as physical quantities for an isotropic fluid in motion

$$T^{\alpha\beta} + \tfrac{1}{2}T\gamma^{\alpha\beta} = \frac{8\pi G}{c^4}\left[\frac{(p + \rho c^2)v^\alpha v^\beta}{c^2 - v^2} + \tfrac{1}{2}(\rho c^2 - p)\gamma^{\alpha\beta}\right].$$

It should be stressed that all these equations hold good even when space-time is strongly curved. Unlike some treatments they are not restricted to nearly flat-space but it is assumed that the space-time is stationary.

To find the general spherically symmetric solution for empty space we take $dl^2 = \gamma_{\alpha\beta}dx^\alpha dx^\beta = e^{2\lambda}dr^2 + r^2 d\hat{\mathbf{r}}^2$ where $\hat{\mathbf{r}}$ is the unit Cartesian vector $(\sin\theta\cos\phi, \sin\theta\sin\phi,\ \cos\theta)$. Then $d\hat{\mathbf{r}}^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ but the advantage of the vector notation is that no axis for θ, ϕ need be taken. In spherical symmetry $\mathbf{B}_g$ must be radial and divergenceless so Gauss's theorem gives $|\mathbf{B}_g|r^2 = Q =$ const which is the field of a gravomagnetic monopole

$$B_g^r = -Qe^{-\lambda}/r^2. \tag{3.11}$$

Reinserting $\mathbf{E}_g = c^2\nabla\nu$ into (3.8) we have

$$R_{00} = -\nu'' + \nu'^2 - 2\nu'/r + \lambda'\nu' + \tfrac{1}{2}e^{2(\lambda-\nu)}Q^2(cr)^{-4} = 0. \tag{3.12}$$

To form $P^{\alpha}{}_{\beta}$ we need the 3 dimensional Christofel symbols

$$\lambda^{\sigma}_{\mu\nu} = \tfrac{1}{2}\gamma^{\sigma\eta}(\gamma_{\eta\mu,\nu} + \gamma_{\eta\nu,\mu} - \gamma_{\mu\nu,\eta}) \tag{3.13}$$

which are

$$\begin{aligned}
&\lambda^{\sigma}_{\mu\sigma} = \tfrac{1}{2\gamma}\gamma_{,\mu} && \lambda^{\sigma}_{\phi\phi} = -\tfrac{1}{2}\gamma^{\sigma\eta}\gamma_{\phi\phi,\eta} \\
&\lambda^{\sigma}_{rr} = \begin{cases} 0 & \sigma\neq r \\ \frac{1}{2}\gamma^{rr}\gamma_{rr,r} & \sigma=r \end{cases} && \lambda^{\sigma}_{\theta\theta} = \begin{cases} 0 & \sigma\neq r \\ -\frac{1}{2}\gamma^{rr}\gamma_{\theta\theta,r} & \sigma=r \end{cases} \\
&\lambda^{\sigma}_{\phi\sigma} = 0 && \lambda^{\sigma}_{r\tau} = \tfrac{1}{2}\gamma^{\sigma\eta}\gamma_{\eta\tau,r}
\end{aligned} \tag{3.14}$$

(3.6), (3.7) and (3.9) are identically satisfied. The surviving equations of (3.10) are

$$R^{rr} = -\nu'' + \nu'^2 + \lambda'\nu - 2\lambda'/r = 0 \tag{3.15}$$

and

$$\begin{aligned} R^{\theta\theta} = R^{\phi\phi} = \lambda' e^{-2\lambda} - \frac{e^{-2\lambda}}{r} + \frac{1}{r} + \frac{1}{2}e^{-2\nu}Q^2c^{-4}r^{-3} + \\ + \nu' e^{-2\lambda} = 0. \end{aligned} \tag{3.16}$$

Equations (3.12), (3.15) and (3.16) must be solved for ν and λ. Eliminating ν'' from (3.12) and (3.15) we find

$$2(\lambda' - \nu') + \tfrac{1}{2}e^{2(\lambda-\nu)}Q^2c^{-4}r^{-3} = 0 \tag{3.17}$$

which integrates on division by $e^{2(\lambda-\nu)}$ giving

$$e^{-2(\lambda-\nu)} = -q^2r^{-2} + C \tag{3.18}$$

where $q = Q/2c^2$ which has the dimensions of a length. Multiplying (3.16) by $e^{2\nu}$ and using (3.17) and (3.18) we have

$$(C - q^2r^{-2})(r^{-1} - 2\nu') = q^2r^{-3} + e^{+2\nu}/r$$

dividing by $e^{2\nu}(C - q^2r^{-2})$ we obtain

$$(e^{-2\nu})' + \frac{1}{r}\left(\frac{Cr^2 - 2q^2}{Cr^2 - q^2}\right)e^{-2\nu} - \frac{r}{Cr^2 - q^2} = 0$$

which is linear in $e^{-2\nu}$ and readily solved by integrating factor to give

$$e^{-2\nu} = \frac{1}{C} - \frac{2q^2}{Cr^2} + \frac{2\overline{C}}{r^2}\sqrt{Cr^2 - q^2} \tag{3.19}$$

where C and $\overline{C}$ are both constants. It follows from (3.18)

$$\gamma_{rr} = e^{2\lambda} = (C - q^2 r^{-2})^{-1} e^{+2\nu}. \tag{3.20}$$

To get $e^{-2\nu}$ and $\gamma_{\alpha\beta}$ asymptotically of Schwarzschild form we need $C = 1$ and $\overline{C} = -\widetilde{m}$, the asymptotic mass GM/c^2. Thus we find

$$g_{00} = e^{-2\nu} = 1 - 2r^{-2}\left(q^2 + \widetilde{m}\sqrt{r^2 - \ell^2}\,\right) \tag{3.21}$$

$$\gamma_{rr} = (1 - q^2 r^{-2})e^{+2\nu} \tag{3.22}$$

which are the metric components of NUT space. Notice that when $Q = 2qc^2 = 0$ this reduces to Schwarzschild's metric. The metric is completed by taking a vector potential A_α for the gravomagnetic field $\mathbf{B}_g$; any one will do since they are connected by gauge transformation which merely changes the zero point of time. As we saw in Section I it is impossible to choose a spherically symmetric vector potential but this does not affect the spherical symmetry of the physics. A suitable vector potential is that given in (1.19) which gives us the metric

$$\begin{aligned} ds^2 = {} & e^{-2\nu}\left(cdt - 2q(1 + \cos\theta)d\phi\right)^2 - \\ & - (1 - q^2/r^2)e^{+2\nu}dr^2 - r^2 d\hat{\mathbf{r}}^2 \end{aligned} \tag{3.23}$$

where $e^{-2\nu}$ is given by (3.21). This metric is more commonly written in terms of the radial variable $\tilde{r} = \sqrt{r^2 - q^2}$ because the square roots disappear leaving an analytic expression, however we have preferred the variable that makes the surface area of the sphere $4\pi r^2$ as in Schwarzschild space. Of course the metric (3.23) appears to have a preferred axis but this is illusory because we can switch it into any direction we like by a gauge transformation, see the discussion under (1.20). The horizon where g_{00} changes sign is given by $\tilde{r} = \widetilde{m} + \sqrt{q^2 + \widetilde{m}^2}$ at which point γ_{rr} changes sign also as in Schwarzschild space.

NUT-Space was discovered in Ehlers' thesis (1957) and rediscovered by Newman, Tamburino & Unti (1963). It is closely related to Taub's (1951) metric and their relationship has been beautifully illuminated by Misner & Taub (1969). The fact that NUT space has a gravomagnetic monopole was found by Demianski & Newman (1966), who also found a NUT version of Kerr space. See also Dowker & Roche (1967).

B. Orbits and Gravitational Lensing by NUT Space

The geodesics of NUT space may be determined from $\delta \int ds = 0$ using the metric in the form (3.1). When $ds^2 \neq 0$ we write τ for the proper time and when $ds^2 \neq 0$ we replace it by an affine parameter (which we also call τ). Varying $\dot{t} = dt/d\tau$ and using the fact that the metric is stationary we have

$$e^{-2\nu}(c\dot{t} - A_\alpha \dot{x}^\alpha) = \varepsilon = \text{ constant.} \tag{3.24}$$

Varying x^α we find

$$\begin{aligned} \delta x^\alpha \Bigg\{ \frac{d}{d\tau} \left[e^{-2\nu}(c\dot{t} - A_\beta \dot{x}^\beta) A_\alpha + \gamma_{\alpha\beta}\dot{x}^\beta \right] + \\ + \tfrac{1}{2} \frac{\partial}{\partial x^\alpha} \left[e^{-2\nu}(c\dot{t} - A_\beta \dot{x}^\beta)^2 - \gamma_{\beta\gamma}\dot{x}^\beta \dot{x}^\gamma \right] \Bigg\}. \end{aligned} \tag{3.25}$$

Using (3.24) and transferring the $\varepsilon dA_\alpha/d\tau = \varepsilon \dot{x}^\beta \partial_\beta A_\alpha$ term into the second bracket we find the equation of motion in which $\mathbf{A}$ only occurs through $\partial_\alpha A_\beta - \partial_\beta A_\alpha = \eta_{\alpha\beta\gamma} B^\gamma$ where $\eta_{\alpha\beta\gamma}$ is the antisymmetric tensor, $\sqrt{\gamma}$ times the alternating symbol.

$$\begin{aligned} \delta x^\alpha \Bigg[\frac{d}{d\tau}(\gamma_{\alpha\beta}\dot{x}^\beta) + \tfrac{1}{2} \frac{\partial}{\partial x^\alpha}(e^{-2\nu})\varepsilon^2 e^{+4\nu} - \\ - \tfrac{1}{2}\gamma_{\beta\gamma,\alpha}\dot{x}^\beta \dot{x}^\gamma - \varepsilon \eta_{\alpha\beta\gamma} B^\gamma \dot{x}^\beta \Bigg] = 0 \end{aligned} \tag{3.26}$$

We now write $\gamma_{\alpha\beta}$ in the form involving the unit Cartesian vector $\hat{\mathbf{r}}$,

$$\gamma_{\alpha\beta} dx^\alpha dx^\beta = e^{2\lambda} dr^2 + r^2 (d\hat{\mathbf{r}})^2.$$

$\delta\hat{\mathbf{r}}$ the variation of $\hat{\mathbf{r}}$ is an arbitrary small vector perpendicular to $\hat{\mathbf{r}}$. Thus making variations with r fixed we deduce from (3.26) Using (3.11) for B_g^r

$$\delta\hat{\mathbf{r}} \cdot \left[\frac{d}{d\tau}\left(r^2\frac{d\hat{\mathbf{r}}}{d\tau}\right) + \varepsilon\frac{d\hat{\mathbf{r}}}{d\tau} \times Q\hat{\mathbf{r}}\right] = 0$$

since $\delta\hat{\mathbf{r}}$ is an arbitrary vector perpendicular to $\hat{\mathbf{r}}$ we deduce that

$$\hat{\mathbf{r}} \times \frac{d}{dt}\left(r^2\frac{d\hat{\mathbf{r}}}{d\tau}\right) = \frac{d\mathbf{L}}{d\tau} = -\frac{d}{d\tau}\left(\varepsilon Q\hat{\mathbf{r}}\right)$$

$$\mathbf{L} + \varepsilon Q\hat{\mathbf{r}} = \mathbf{j} = \text{ const.} \tag{3.27}$$

Except for the factor ε which reduces to m_0c^2 in the non-relativistic case we see that this is precisely the vector integral (1.14). Dotting Eq. (3.27) with $\hat{\mathbf{r}}$ we find $\mathbf{j}\cdot\hat{\mathbf{r}} = \varepsilon Q$ showing that $\hat{\mathbf{r}}$ lies on a cone similarly $\mathbf{L}\cdot\mathbf{j} = L^2 = \mathbf{j}^2 - \varepsilon^2Q^2 =$ const so $\mathbf{L}$ moves around a cone. The radial equation of motion is redundant since we may use the energy and the equation $(ds/d\tau)^2 = U = 1$ or 0 instead. U is 1 for time like geodesics and 0 for light-like ones.

This gives us

$$\varepsilon^2e^{+2\nu} - \dot{r}^2e^{2\lambda} - L^2r^{-2} = U. \tag{3.28}$$

To see the geometry of the trajectory we introduce the curvilinear angle φ of §1 measured around the cone's surface. Then $r^2\dot{\varphi} = L$ so Eq. (3.1) can be integrated by quadrature

$$\varphi - \varphi_0 = \int \frac{Lr^{-2}dr}{\sqrt{\varepsilon^2e^{-2(\lambda-\nu)} - (U + L^2r^{-2})e^{-2\lambda}}}. \tag{3.29}$$

In general this integral can not be performed explicitly for the λ and ν of NUT space even after substitution in terms of $\tilde{r}$ to make it more analytic. We therefore turn to the $r^2 \gg q^2 + \widetilde{m}^2$ limit well away from the event horizon. This is the important case in all gravitational lenses observed to date. In that limit the q^2/r^2 term in the effective potential is attractive and therefore of the wrong sign to give the non-precessing orbits of Section I. The precession around the cones is faster than in the classical Kepler + monopole problem by a factor 3/2. To the first order in $\widetilde{m}/b$ where b is the impact parameter, we find a bending angle measured like φ of $\Delta\varphi = 4\widetilde{m}/b$ just as for the Schwarzschild metric; however the difference is that this angle is measured around a cone not in a plane. Again to first order we can find the effect of the gravomagnetic field by integrating the momentum transfer along the unperturbed straight line path. This gives an out-of-plane bending of $4q/b$; a result that is confirmed by the full NUT space calculation. Nouri-Zonoz, M. & Lynden-Bell, D. (1997). Thus the major effect of the gravomagnetic monopole Q is to twist the rays that pass it. While the bending angle is proportional to b^{-1}, the effect is exaggerated when looking down the line toward the NUT lens by the factor D_L/b, so the twist around the lens is $4qD_L/b^2$. Here D_L is the distance from the observer to the lens. The same exaggeration factor occurs for the normal gravitational bending so for a source at infinity and an image at (b, θ) in the plane of the sky at the lens's distance, the apparent position of the source is

$$(b_s, \theta_s) = \left(b \left(1 - \frac{4\widetilde{m} D_L}{b^2} + \frac{8q^2 D_L^2}{b^4} \right), \theta - \frac{4qD_L}{b^2} \right).$$

This expression defines a map from image to source. From this map one can work out both the shear and the magnification of a NUT lens in the large impact parameter régime. The magnification of area and thus luminosity is

$$db^2/db_s^2 = [1 - 16b^{-4} D_L^2 (m^2 + q^2)]^{-1}. \tag{3.30}$$

A small circular source will be imaged into an ellipse of axial ratio

$$\frac{b^2 + 4\ D_L(m + \sqrt{m^2 + q^2})}{b^2 + 4\ D_L(m - \sqrt{m^2 + q^2})}$$

with the short axis of the ellipse inclined to the radius at the angle (see Fig. 9)

$$\tan^{-1} \left(\frac{q}{m + \sqrt{m^2 + q^2}} \right).$$

This is 45° for $q \gg m$ and 13° for $Q = 2qc^2 = mc^2$. This spiral conformation of the images about a NUT lens is very characteristic. It is not displayed in normal gravitational imaging and the gravomagnetic lens due to a rotating object seen pole on does not show it because the twist of the ray as it approaches such a lens is cancelled by the opposite twist as it recedes. Thus the discovery of a spiral shear field about a lens would indicate the presence of a gravomagnetic monopole. Such effects should be looked for by those studying gravitational lenses. The expectation must be small but the reward might be an amazing discovery.

C. Quantization of Gravomagnetic Monopoles and their Classical Physics

By analogy with Dirac's argument for the quantization of magnetic monopoles and charges, Dowker & Roche (1967), Dowker (1974), Hawking (1979) and Zee (1985) have suggested quantization of gravomagnetic monopoles and energy. Corresponding to Dirac's $Q_m e = \frac{1}{2}N\hbar c$ for magnetic Q_m, they have $Qm_0 = \frac{1}{2}N\hbar c$ for gravomagnetic monopole Q. This implies that both Q and mass m_0 are quantized in conjugate units Q_1 and m_1 obeying $Q_1 m_1 = \frac{1}{2}\hbar c$. Whereas such ideas are naturally attractive they do not naturally lead to a self-consistent relativistic theory. For instance, looking at the Klein-Gordon equation in NUT space and separating variables with $\psi \propto e^{i(m\phi+\omega t)}$, one finds an eigenvalue equation for ω. The Dirac monopole quantization condition, $Q(\hbar\omega/c^2) = \frac{1}{2}N\hbar c$ with N an integer, shows us that the only possible eigenvalues ω are integer multiples of $\frac{1}{2}c^3/Q$ and the corresponding energy $\hbar\omega$ is the total energy of the 'orbit' including rest mass. However this condition conflicts with the energies of the bound states[2] which are not integer multiples of any unit. Mueller & Perry (1986). Thus if such ideas are viable at all a more radical change in basic theory is needed. In $+++-$ NUT space it does not appear to be possible to build a consistent quantum theory like Dirac's magnetic monopole theory. This is what Ross (1983) concluded and is related to Misner's (1963) finding that NUT space contains closed timelike lines, with time being periodic every $8\pi q/c$. For a discussion of the energy levels in $++++$ Taub-NUT space, the reader is referred to the papers by Gibbons & Manton (1986). This space was shown to be relevant to the interactions of monopoles by Atiyah

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[2]To get definite bound states one must impose a potential barrier so that the black hole is not reached.

& Hitchin (1985). §If magnetic monopoles exist, Maxwell's equations must be changed to include div $\mathbf{B} = 4\pi\rho_m$, Curl $\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 4\pi \mathbf{j}_m$, where ρ_m is the monopole density and $\mathbf{j}m$ is the monopole current density. Such modified Maxwell equations do not come with a vector potential. It is natural to ask how general relativity must be modified to allow for gravomagnetic monopole densities and currents. While this is not so obvious we conjecture the generalization will be to spaces with unsymmetric affine connections which have non-zero torsion. It would be interesting to demonstrate this conjecture as it could introduce a greater degree of physical understanding of those spaces.

IV. OBSERVABILITY

Following Kibble's (1980) suggestion that magnetic monopoles would be a natural consequence of the Big Bang, they have long been sought.

We have concentrated on the spectra of monopolar atoms and the lensing properties of gravomagnetic monopoles since these are ways in which, at least in principle, monopoles might be discovered observationally. Spectroscopically one may argue that the best place to look is in the spectra of supernovae, quasars or active galactic nuclei where the basic Ly α lines of Table I or II might be seen as very weak absorption lines in very high resolution spectra. Quasars have the advantage that these lines will be shifted into the visible. We have looked at IUE ultra-violet spectra of Supernova 1987A and seen no lines at the wavelengths 2774.62 or 2733.78. More supernovae and stacked high resolution spectra of quasars should be pursued. Although in regions of observed magnetic fields the limits obtained spectroscopically will fall far short of the Parker (1970) bound. While the nature of the dark matter that constitutes most mass in the universe remains unknown, such esoteric possibilities are worth pursuit.

Searches on Earth have produced one unrepeatable event and a monopolar observatory under the Grand Sasso that has so far found no monopoles in Cosmic Rays. There has been a speculative suggestion, Kephart & Weiler (1996), that the leveling up on the numbers of cosmic rays at the highest energies might be due to monopoles but there is no confirmation of that idea. To date the best limit on the numbers of monopoles in interstellar space comes from the Parker (1970-71) bound. This arises from the idea that too many magnetic monopoles would 'short out' the galactic magnetic fields that are observed. A good general discussion of such limits may be found in the book of Kolb & Turner (1991). For more recent work on monopoles in field theory see reviews by Olive (1996, 97), and the papers by Sen (1994) and by Seiberg & Witten (1994). More details of the fundamental work on monopoles in field theory by 't Hooft (1974) and by Polyakov (1974) can be found in the review by Goddard & Olive.

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FIGURES

FIG. 1. The circular orbits about a central potential endowed with a monopole. The orbits in opposite senses are displaced above and below the centre of force. For a Newtonian potential their vertical separation gradually increases as their radii are increased. The orbits with a given angular momentum $|\mathbf{L}|$ lie on cones with opening angle $\cos^{-1}(Q_*/|\mathbf{j}|)$.

FIG. 2. When one of the cones is slit and flattened a gap opens along the slit. On the cone itself the sides of this gap are identified. Orbits which close on a plane will not close on the cone because of the gap. As a result they precess.

FIG. 3. An ellipse precessing around a cone of semi angle $70^{\rm o}$ making a rosette orbit on it.

FIG. 4. A monopole and its $\mathbf{B}$ field showing the surfaces S_1, S_2 and $S_3 \equiv S_1 - S_2$.

FIG. 5. j values allowed by the conditions $j \geq \frac{1}{2}(|m| + |m - \frac{N}{2}|)$. j cannot be less than the average of the two faint V lines in the diagram.

FIG. 6. Energy levels for a spinning electron in hydrogen with 0, 1, 2 or 3 Dirac monopoles on its nucleus. Excepting 'isotopic' shifts due to the changed nuclear mass and relativistic corrections, the energy levels of the ground states are in the ratio $1 : \frac{1}{2} : \frac{1}{3} : \frac{1}{4}$.

FIG. 7. Energy level diagram $E(n, J)$ for $N = 1$, hydrogen with one Dirac monopole on its nucleus. The nucleus has been assumed to be fixed.

FIG. 8. Energy level diagram $E(n, J)$ for $N = 2$, hydrogen with two Dirac monopoles on its nucleus.

FIG. 9. Gravitational Lensing by NUT space of a small circular source at S appears as an inclined ellipse at the image I. Many such images make a spiral effect around the NUT lens L.

TABLES

TABLE I. Wavelengths in Angstroms of the two Lyman Series, the five Balmer Series and the eight Paschen Series of hydrogen with $(N = 1)$ one Dirac monopole attached to the proton cf. Fig. 7. The wavelengths after the dots are those of the series limits.

TABLE II. Wavelengths in Angstroms of the Lyman, Balmer and Paschen of hydrogen with two Dirac monopoles attached to the proton cf. Fig. 8 and 6.