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THE EXACT MOTION OF A CHARGED PARTICLE IN THE MAGNETIC FIELD

$$\boldsymbol{B} = (x^2 + y^2)^{-\frac{1}{2}} \left(\frac{-\gamma y}{x^2 + y^2}, \frac{\gamma x}{x^2 + y^2}, \alpha \right)$$

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Synopsis

The relativistic equations of motion are completely solved for a special configuration of the external magnetic field, a case which seems to have never been considered before. It is also shown how the quaternionic formalism can replace the traditional Lagrange formalism in that kind of problem. The characteristics of the movement are briefly discussed in all possible cases.

1. The equations of motion of a charged particle in an electromagnetic field are rarely exactly integrable[1]). We present here the complete solution of the problem for the special magnetic field

$$\boldsymbol{B} = (x^2 + y^2)^{-\frac{1}{2}} \left(\frac{-\gamma y}{x^2 + y^2}, \frac{\gamma x}{x^2 + y^2}, \alpha \right).$$

This case seems to be isolated. The same problem has been solved by us in Dirac's theory[2]).

Let us first remark that there is no difference between the classical treatment or the relativistic one. Indeed when there is only a magnetic field, the relativistic equations can be written

$$m_0\beta\boldsymbol{\gamma} + m_0\beta^3(\boldsymbol{v}\cdot\boldsymbol{\gamma}/c^2)\,\boldsymbol{v} = \varepsilon\boldsymbol{v} \times \boldsymbol{B}, \tag{1}$$

where

$$\beta = (1 - v^2/c^2)^{-\frac{1}{2}}.$$

Multiplying by $\boldsymbol{v}$, it follows that $\boldsymbol{v}\cdot\boldsymbol{\gamma} = 0$. Therefore

$$v^2 = \text{const.} \tag{2}$$

Eq. (1) then reduces to the newtonian form:

$$m_0\beta\boldsymbol{\gamma} = \varepsilon\boldsymbol{v} \times \boldsymbol{B} = m\boldsymbol{\gamma},$$

where m is the relativistic mass.

We may therefore start with this last equation.

The use of cylindrical coordinates (r, φ, z) is needed; the equations of motion may be obtained through a Lagrange formalism or otherwise by using the quaternionic formalism (for definitions related to quaternions see for example ref. 3). The starting equation may be written

$$\frac{\mathrm{d}^2R_3}{\mathrm{d}t^2} = \frac{\varepsilon}{m}\left[\mathrm{Vect}\,\frac{\mathrm{d}R_3}{\mathrm{d}t}\,B_3\right], \tag{3}$$

where we have set

$$R_3 = ix + jy + kz = r\,\mathrm{i}\,\mathrm{e}^{-k\varphi} + kz,$$

$$B_3 = iB_x + jB_y + kB_z = \mathrm{e}^{\frac{1}{2}k\varphi}\left(j\,\frac{\gamma}{r^2} + k\,\frac{\alpha}{r}\right)\mathrm{e}^{-\frac{1}{2}k\varphi}.$$

Introducing these expressions into (3) we get

$$\mathrm{e}^{\frac{1}{2}k\varphi}\left\{i\left[\frac{\mathrm{d}^2r}{\mathrm{d}t^2} - r\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2\right] + j\left(r\,\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} + 2\,\frac{\mathrm{d}r}{\mathrm{d}t}\,\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right) + k\,\frac{\mathrm{d}^2z}{\mathrm{d}t^2}\right\}\mathrm{e}^{-\frac{1}{2}k\varphi}$$
$$= \frac{\varepsilon}{m}\,\mathrm{e}^{\frac{1}{2}k\varphi}\left[\mathrm{Vect}\left(i\,\frac{\mathrm{d}r}{\mathrm{d}t} + jr\,\frac{\mathrm{d}\varphi}{\mathrm{d}t} + k\,\frac{\mathrm{d}z}{\mathrm{d}t}\right)\left(j\,\frac{\gamma}{r^2} + k\,\frac{\alpha}{r}\right)\right]\mathrm{e}^{-\frac{1}{2}k\varphi}.$$

That quaternionic equation is equivalent to the following system:

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2} - r\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 = \frac{\varepsilon}{m}\left[\alpha\,\frac{\mathrm{d}\varphi}{\mathrm{d}t} - \frac{\gamma}{r^2}\,\frac{\mathrm{d}z}{\mathrm{d}t}\right], \tag{4}$$

$$2\,\frac{\mathrm{d}r}{\mathrm{d}t}\,\frac{\mathrm{d}\varphi}{\mathrm{d}t} + r\,\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} = -\,\frac{\varepsilon}{m}\,\frac{\alpha}{r}\,\frac{\mathrm{d}r}{\mathrm{d}t}, \tag{5}$$

$$\frac{\mathrm{d}^2z}{\mathrm{d}t^2} = \frac{\varepsilon}{m}\,\frac{\gamma}{r^2}\,\frac{\mathrm{d}r}{\mathrm{d}t}. \tag{6}$$

Eq. (4) may be replaced by (2) *i.e.*

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + r^2\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2 = v^2. \tag{7}$$

Eqs. (5) and (6) are immediately equivalent to conservation laws:

$$r^2\,\frac{\mathrm{d}\varphi}{\mathrm{d}t} + \frac{\varepsilon\alpha}{m}\,r = b \qquad (= \text{const.}), \tag{8}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{\varepsilon\gamma}{m}\,\frac{1}{r} = a \qquad (= \text{const.}). \tag{9}$$

Combining (7), (8) and (9) we reduce the solution of our problem to the following basic problems:

$$t = \int_{r_0}^{r} \frac{u\,\mathrm{d}u}{(Au^2 + Bu + C)^{\frac{1}{2}}}, \tag{10}$$

$$\varphi - \varphi_0 = \int_{r_0}^{r} \frac{[b - (\varepsilon\alpha/m)\,u]\,\mathrm{d}u}{u(Au^2 + Bu + C)^{\frac{1}{2}}}, \tag{11}$$

$$z - z_0 = \int_{r_0}^{r} \frac{[(au - (\varepsilon\gamma/m)]\,\mathrm{d}u}{(Au^2 + Bu + C)^{\frac{1}{2}}}. \tag{12}$$

We have put for the sake of brevity:

$$\begin{aligned} A &= (m^2v^2 - \varepsilon^2\alpha^2 - a^2m^2)/m^2, \qquad B = 2m\varepsilon(b\alpha + a\gamma)/m^2,\\ C &= -(b^2m^2 + \varepsilon^2\gamma^2)/m^2,\\ B^2 - 4AC &= (4/m^4)[m^2v^2(\varepsilon^2\gamma^2 + m^2b^2) - (abm^2 - \varepsilon^2\alpha\gamma)^2]. \end{aligned} \tag{13}$$

It may be noticed that $B^2 - 4AC \geq 0$. That is evident when $A \geq 0$ because C is always negative. When $A < 0$ that is also true because

$$Ar^2 + Br + C = r^2\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 \geq 0.$$

and this is possible if and only if the trinomial has two real roots r_1 and r_2 so that $r_1 < r < r_2$.

Before discussing the various cases depending upon the values A and C, let us first point out the following equations. Setting

$$I(u) = \int \frac{u\,\mathrm{d}u}{(Au^2 + Bu + C)^{\frac{1}{2}}}, \qquad J(u) = \int \frac{\mathrm{d}u}{(Au^2 + Bu + C)^{\frac{1}{2}}},$$

$$K(u) = \int \frac{\mathrm{d}u}{u(Au^2 + Bu + C)^{\frac{1}{2}}},$$

we deduce

$$\begin{aligned} t &= I(r) - I(r_0),\\ z - z_0 &= aI(r) - \frac{\varepsilon\gamma}{m} J(r) - aI(r_0) + \frac{\varepsilon\gamma}{m} J(r_0),\\ \varphi - \varphi_0 &= bK(r) - \frac{\varepsilon\alpha}{m} J(r) - bK(r_0) + \frac{\varepsilon\alpha}{m} J(r_0). \end{aligned} \tag{14}$$

The following five possible cases are to be successively investigated

I–1:	$C = 0$	$A = 0$;
I–2:	$C = 0$	$A > 0$;
II–1:	$C < 0$	$A = 0$;
II–2:	$C < 0$	$A > 0$;
II–3:	$C < 0$	$A < 0$.

In case I–2 $A < 0$ is impossible when $C = 0$ and will therefore never appear.

2. *The solutions*. I–1: $C = 0$, $A = 0$.

$C = 0$ implies $b = \gamma = 0$ and therefore $B = 0$ *i.e.* $\mathrm{d}r/\mathrm{d}t = 0$; we deduce $r = r_0\,(= \text{const.})$, $z = z_0 + at$ and $\varphi = \varphi_0 - (\varepsilon\alpha/mr_0)\,t$.

I–2: $C = 0$, $A > 0$.

$\mathrm{d}r/\mathrm{d}t = A^{\frac{1}{2}}$ implies $r = r_0 + A^{\frac{1}{2}}t$, $z = z_0 + at$ and

$$\varphi = \varphi_0 - \frac{\varepsilon\alpha A^{-\frac{1}{2}}}{m} \ln \frac{r_0 + A^{\frac{1}{2}}t}{r_0}.$$

II–1. $C < 0$, $A = 0$.

The calculation of integrals I, J and K give on account of (14):

$$I(u) = \frac{2}{3B^2}(Bu + C)^{\frac{3}{2}} - \frac{2C}{B^2}(Bu + C)^{\frac{1}{2}},$$

$$J(u) = \frac{2}{B}(Bu + C)^{\frac{1}{2}},$$

$$K(u) = 2(-C)^{-\frac{1}{2}} \operatorname{arctg}\left[\frac{Bu + C}{-C}\right]^{\frac{1}{2}}.$$

$$t = \frac{2}{3B^2}(Br + C)^{\frac{3}{2}} - \frac{2C}{B^2}(Br + C)^{\frac{1}{2}} - I(r_0),$$

$$z - z_0 = \frac{2a}{3B^2}(Br + C)^{\frac{3}{2}} - \\ - \frac{2amC + 2B\varepsilon\gamma}{mB^2}(Br + C)^{\frac{1}{2}} + \frac{\varepsilon\gamma}{m} J(r_0) - aI(r_0), \tag{15}$$

$$\varphi - \varphi_0 = 2b(-C)^{-\frac{1}{2}} \operatorname{arctg}\left[\frac{Br + C}{-C}\right]^{\frac{1}{2}} \\ - \frac{2\varepsilon\alpha}{mB}(Br + C)^{\frac{1}{2}} + \frac{\varepsilon\alpha}{m} J(r_0) - bK(r_0).$$

II–2: $C < 0$, $A > 0$.

$$
\begin{aligned}
I(u) &= \frac{1}{A}\,(Au^2 + Bu + C)^{\frac{1}{2}} \\
&\quad - \frac{B}{2}\,A^{-\frac{3}{2}} \ln\,|2A^{\frac{1}{2}}(Au^2 + Bu + C)^{\frac{1}{2}} + 2Au + B|, \\
J(u) &= A^{-\frac{1}{2}} \ln|2A^{\frac{1}{2}}\,(Au^2 + Bu + C)^{\frac{1}{2}} + 2Au + B|, \\
K(u) &= (-C)^{-\frac{1}{2}} \arcsin \frac{Bu + 2C}{u(B^2 - 4AC)^{\frac{1}{2}}}\,.
\end{aligned}
$$

$$
\begin{aligned}
t &= \frac{1}{A}\,(Ar^2 + Br + C)^{\frac{1}{2}} \\
&\quad - \frac{B}{2}\,A^{-\frac{3}{2}} \ln|2A^{\frac{1}{2}}(Ar^2 + Br + C)^{\frac{1}{2}} + 2Ar + B| - I(r_0),
\end{aligned}
$$

$$
\begin{aligned}
z - z_0 &= \frac{a}{A}\,(Ar^2 + Br + C)^{\frac{1}{2}} \\
&\quad - \frac{amB + 2\varepsilon\gamma A}{2mA^{\frac{3}{2}}} \ln|2A^{\frac{1}{2}}\,(Ar^2 + Br + C)^{\frac{1}{2}} + 2Ar + B| \\
&\quad - aI(r_0) + \frac{\varepsilon\gamma}{m}\,J(r_0),
\end{aligned}
$$

$$
\begin{aligned}
\varphi - \varphi_0 &= b(-C)^{-\frac{1}{2}} \arcsin \frac{Br + 2C}{r(B^2 - 4AC)^{\frac{1}{2}}} \\
&\quad - \frac{\varepsilon\alpha A^{-\frac{1}{2}}}{m} \ln|2A^{\frac{1}{2}}\,(Ar^2 + Br + C)^{\frac{1}{2}} + 2Ar + B| \\
&\quad - bK(r_0) + \frac{\varepsilon\alpha}{m}\,J(r_0).
\end{aligned}
$$

II–3: $C < 0$, $A < 0$.

$$
\begin{aligned}
I(u) &= \frac{1}{A}\,(\mathrm{A}u^2 + Bu + C)^{\frac{1}{2}} \\
&\quad - \frac{B}{2}\,(-A)^{-\frac{3}{2}} \arcsin \frac{2Au + B}{(B^2 - 4AC)^{\frac{1}{2}}}, \\
J(u) &= -(-A)^{-\frac{1}{2}} \arcsin \frac{2Au + B}{(B^2 - 4AC)^{\frac{1}{2}}}, \\
K(u) &= (-C)^{-\frac{1}{2}} \arcsin \frac{Bu + 2C}{u(B^2 - 4AC)^{\frac{1}{2}}}.
\end{aligned}
$$

$$t = \frac{1}{A}(Ar^2 + Br + C)^{\frac{1}{2}} - \frac{B}{2}(-A)^{-\frac{3}{2}} \arcsin \frac{2Ar + B}{(B^2 - 4AC)^{\frac{1}{2}}} - I(r_0),$$

$$z - z_0 = \frac{a}{A}(Ar^2 + Br + C)^{\frac{1}{2}} - \frac{aBm + 2\varepsilon\gamma A}{2m(-A)^{\frac{3}{2}}} \arcsin \frac{2Ar + B}{(B^2 - 4AC)^{\frac{1}{2}}} - aI(r_0) + \frac{\varepsilon\gamma}{m} J(r_0),$$

$$\varphi - \varphi_0 = b(-C)^{-\frac{1}{2}} \arcsin \frac{Br + 2C}{r(B^2 - 4AC)^{\frac{1}{2}}} + \frac{\varepsilon\alpha}{m(-A)^{\frac{1}{2}}} \arcsin \frac{2Ar + B}{(B^2 - 4AC)^{\frac{1}{2}}} - bK(r_0) + \frac{\varepsilon\alpha}{m} J(r_0).$$

3. *Study of the trajectories. Summary.* From the analytical point of view the problem is completely solved; a, b and v^2 are related to the initial conditions by eqs. (7), (8) and (9). Let us now review in a little more detail the five cases and let us try to interpret the general characteristics of the movement at least when it is not too complicated.

I–1. This case occurs when

$$\gamma = 0,\ r_0\left(\frac{d\varphi}{dt}\right)_0 = -\frac{\varepsilon\alpha}{m} \qquad (i.e.\ C = 0),$$

and

$$m^2v^2 = \varepsilon^2\alpha^2 + m^2\left(\frac{dz}{dt}\right)_0^2 \qquad (i.e.\ A = 0).$$

The trajectory is a helix; its projection on the xy plane is a circle of radius r_0. The particle travels to infinity along the z axis with the constant velocity $a = (dz/dt)_0$.

I–2. This case occurs when

$$\gamma = 0,\ r_0\left(\frac{d\varphi}{dt}\right)_0 = -\frac{\varepsilon\alpha}{m} \qquad (i.e.\ C = 0),$$

and

$$m^2v^2 > \varepsilon^2\alpha^2 + m^2\left(\frac{dz}{dt}\right)_0^2 \qquad (i.e.\ A > 0).$$

The trajectory is a helical spiral; its projection on the xy plane is a spiral:

$$\varphi - \varphi_0 = -\frac{\varepsilon\alpha A^{-\frac{1}{2}}}{m} \ln \frac{r}{r_0},$$

or

$$r = r_0 \exp\left[-\frac{mA^{\frac{1}{2}}}{\varepsilon\alpha}(\varphi - \varphi_0)\right],$$

The coordinate r increases at a constant rate $A^{\frac{1}{2}}$. The velocity along the z axis has the constant value $a = (\mathrm{d}z/\mathrm{d}t)_0$.

II–1. This case occurs when γ and $r_0(\mathrm{d}\varphi/\mathrm{d}t)_0 + \varepsilon\alpha/m$ are not simultaneously zero (*i.e.* $C \neq 0$)
and when

$$m^2v^2 = \varepsilon^2\alpha^2 + m^2\left[\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)_0 + \frac{\varepsilon\gamma}{mr_0}\right]^2 \qquad (i.e.\ A = 0).$$

The trajectory is very complicated. However, in the particular case, where $A = 0$ it is possible to invert eq. (15) to obtain r as an explicit function of time; if $X = (Br + C)^{\frac{1}{2}}$ we have to solve the cubic equation

$$X^3 - 3CX - \tfrac{3}{2}B^2\,[t + I(r_0)] = 0.$$

Because C is negative there is only one real root which gives X as a function of time; finally we have

$$r = \frac{1}{B}\left\{\frac{3B^2}{4}[t + I(r_0)] + \left[\frac{9B^4}{16}[t + I(r_0)]^2 - C^3\right]^{\frac{1}{2}}\right\}^{\frac{2}{3}}$$
$$+\frac{1}{B}\left\{\frac{3B^2}{4}[t + I(r_0)] - \left[\frac{9B^4}{16}[t + I(r_0)]^2 - C^3\right]^{\frac{1}{2}}\right\}^{\frac{2}{3}} + \frac{C}{B}.$$

For large values of t we have the asymptotic behaviour

$$r \approx \frac{1}{B}\left(\frac{3B^2t}{2}\right)^{\frac{2}{3}}.$$

II–2 and II–3. These cases occur when γ and $r_0(\mathrm{d}\varphi/\mathrm{d}t)_0 + \varepsilon\alpha/m$ are not simultaneously zero (*i.e.* $C \neq 0$)
and when

$$m^2v^2 > \varepsilon^2\alpha^2 + m^2\left[\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)_0 + \frac{\varepsilon\gamma}{mr_0}\right]^2 \qquad (\text{case II–2},\ A > 0),$$

or

$$m^2v^2 < \varepsilon^2\alpha^2 + m^2\left[\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)_0 + \frac{\varepsilon\gamma}{mr_0}\right]^2 \qquad (\text{case II–3},\ A < 0).$$

In these general cases it is impossible to invert the expression giving t as a function of r. However, it may be noted that in case II–2

$$\frac{-B + (B^2 - 4AC)^{\frac{1}{2}}}{2A} < r < \infty,$$

while in case II–3 the range of allowed r values is bounded:

$$\frac{-B-(B^2-4AC)^{\frac{1}{2}}}{2A} < r < \frac{-B+(B^2-4AC)^{\frac{1}{2}}}{2A},$$

from which we deduce that B is positive in this case because the roots must be positive.

4. *Conclusion.* The problem is now completely solved. Given a set of initial conditions it is always possible to determine which of the five possible cases is at hand. It is then possible to write down the exact solution in terms of elementary functions.

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