A NEW MAGNETIC FIELD FOR DIRAC'S EQUATIONS

A. HAUTOT
University of Liège, Institut de Physique, Sart Tilman, Belgium

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Dirac's equations are exactly solvable for a magnetic field which seems to have never been considered. Energy levels are given.

Many authors are interested in the search for all electric or magnetic fields which lead to exactly solvable Dirac equations [1-3]. We bring a contribution by enunciating that Dirac's equations are also solvable when the electron experiences the following magnetic field (expressed in a cartesian basis $\boldsymbol{B} = \frac{1}{r^3}(-\gamma y, \gamma x, \alpha r^2)$. The corresponding vector potential is: $\boldsymbol{A} = \frac{1}{r}(-\alpha y, \alpha x, \gamma)$. Let us put Dirac's equations into a quaternionic form [4] and solve them by separation of variables [5]:

$$(\nabla_3 + \frac{\epsilon\sqrt{-1}}{\hbar} A_3)u = -u\left(\frac{m_0 c}{\hbar} j + \frac{\sqrt{-1}\,E}{c\hbar} i\right) = -uQ \tag{1}$$

$$\nabla_3 = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} = i\exp(-k\varphi)\frac{\partial}{\partial r} + \frac{1}{r} j\exp(-k\varphi)\frac{\partial}{\partial\varphi} + k\frac{\partial}{\partial z} \tag{2}$$

$$A_3 = iA_x + jA_y + kA_z = \exp(\tfrac{1}{2}k\varphi)\,(j\alpha + k\frac{\gamma}{r})\exp(-\tfrac{1}{2}k\varphi) \tag{3}$$

Introduce eqs. (2) and (3) into (1) and put $u = \exp(\frac{1}{2} k\, \varphi)\, v$:

$$\left[i\left(\frac{\partial}{\partial r} + \frac{1}{2r}\right) + j\left(\frac{1}{r}\frac{\partial}{\partial \varphi} + \frac{\epsilon\sqrt{-1}\alpha}{\hbar}\right) + k\left(\frac{\partial}{\partial z} + \frac{\epsilon\sqrt{-1}\gamma}{\hbar r}\right)\right] v = -v\, Q \tag{4}$$

The separation of variables obviously holds in the form:

$$v = \exp\left(\frac{\sqrt{-1}}{\hbar} p_z z\right) \exp(\sqrt{-1}\,(m + \tfrac{1}{2})\varphi)\, (R_1 + iR_2 + jR_3 + kR_4)$$

The radial equation is

$$ir\frac{\mathrm{d}R}{\mathrm{d}r} + \tfrac{1}{2} iR + j\sqrt{-1}\,(m + \tfrac{1}{2})R + \frac{\sqrt{-1}}{\hbar} p_z\, r\, k\, R + r\, R\, Q + \frac{\epsilon\sqrt{-1}}{\hbar}(j\alpha n + k\gamma) R = 0 \tag{5}$$

equivalent to a coupled differential system with four equations. Note that RQ satisfies the same equation; therefore $RQ = \lambda R$. The eigenvalue λ follows: $\lambda = \pm (E^2 - m_0{}^2 c^4)^{1/2}/c\hbar$. Thus two algebraic relations exist between the four R_i which allows us to eliminate R_1 and R_4. After doing that, make the system real by putting $T_2 = R_2 + R_3\sqrt{-1}$ and $T_3 = R_3 + R_2\sqrt{-1}$ and decouple the system to obtain two second order differential equations. They may be solved by Sommerfeld's polynomial method provided account is taken of our theory [6] for the polynomial solutions of $Df'' + (ar^2 + br + c)f' + (d + er)f = 0$ where D is r, $r(1-r)$ or $r(1-r)(\alpha - r)$. Here $D = r(1-r)$ and the polynomial condition $e = -an$ gives the energy levels while the radial functions contains Laguerre functions only.

$$E^2 = m_0^2 c^4 + c^2 p_z^2 + (\epsilon\,\alpha\, c)^2 - \left[\frac{\epsilon\,\alpha c(2m+1) + 2c\, p_z\, \epsilon\gamma/\hbar}{2n + \sqrt{(2m+1)^2 + 4(\epsilon\gamma/\hbar)^2}}\right]^2$$

References

[1] L. Lam, Phys. Lett. 31 A (1970) 406.
[2] G. N. Stanciu, Phys. Lett. 23 (1966) 232.
[3] Harish-Chandra, Phys. Rev. 74 (1948) 883.
[4] A. W. Conway, Acta Pontificia Academia Scientiarum XII (1948) 259.
[5] A. Hautot, Physica 48 (1970) 609.
[6] A. Hautot, Bull. Soc. Roy. Sc. Liège 38 (1969) 654,
or the general theory see A. Hautot, J. Math. Phys. August of September issue, 1971).

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