SCHRÖDINGER'S EQUATION AND SPECIAL RADIAL ELECTRIC FIELDS

A. P. HAUTOT
University of Liège, Institut de Physique, Sart Tilman par 4000 Liège I, Belgium

Received 14 January 1972

Schrödinger's equation is exactly soluble if one considers a central potential of the type $Ar^2 + Br - D/r$ provided D takes particular well chosen values.

Many authors are interested in the search for all electric fields which allow a complete and exact solution of Schrödinger's equation. Morse and Feshbach [1] have listed convenient potentials: A/r, $A/r - B/r^2$ $(A \neq 0)$, $A/r^2 + Br^2$ $(B \neq 0)$. Some potentials lead to soluble equations provided the angular momentum quantum number $l = 0$: $V = V_0 \exp(-r/d)$ or $V = V_0 \tanh(r/d)$. These cases excepted, Plesset [2] has shown that no exact solution in term of a finite number of elementary functions can be found if the potential is of the type $V = \sum_{k=-m}^{+n} \lambda_k r^k$ with arbitrary constant λ_k. In particular no quantization of energy exists.

Our purpose is to show for the special potential $V = Ar^2 + Br - D/r$ $(A \neq 0)$ that the problem is exactly soluble if D is correctly related to A and B.

1) *Spherical coordinates*: $r^2 = x^2 + y^2 + z^2$. The radial part of Schrödinger's equation is:

$$\psi'' + \frac{2}{r}\psi' - \frac{l(l+1)}{r^2}\psi + \frac{2m}{\hbar^2}\left[E - Ar^2 - Br + \frac{D}{r}\right]\psi = 0, \tag{1}$$

$$\psi = \exp\left[-\frac{1}{2}\sqrt{\frac{2mA}{\hbar^2}}\,r^2 - \frac{1}{2}\sqrt{\frac{2mB^2}{A\hbar^2}}\,r\right] r^{-l-1}\varphi, \tag{2}$$

$$r\varphi'' + \left[-2\sqrt{\frac{2mA}{\hbar^2}}\,r^2 - \sqrt{\frac{2mB^2}{A\hbar^2}}\,r - 2l\right]\varphi' + \left\{l\sqrt{\frac{2mB^2}{A\hbar^2}} + \frac{2mD}{\hbar^2} + \left[\frac{mB^2}{2A\hbar^2} + (2l-1)\sqrt{\frac{2mA}{\hbar^2}} + \frac{2mE}{\hbar^2}\right] r\right\}\varphi = 0. \tag{3}$$

Eq. (3) is of the type $r\varphi'' + (ar^2 + br + c)\varphi' + (d + er)\varphi = 0$ (with $c = -2l$ negative integer). We have studied it elsewhere [3]: the divergent solution (at $r = 0$, see eq. (2)) of eq. (1) is easily found as r^{-l-1} times a linear combination of Weber functions; of course we are mainly interested in the convergent quadratically integrable solution so that $\varphi = r^{2l+1} \times$ polynomial of degree n. Introducing that φ into (3) we easily deduce the two conditions to be fulfilled: $e = -(n + 2l + 1)a$ and the vanishing of a determinant (more precisely a continuant [4]) of range $n + 1$ (thus $n \geq 0$). The detailed form of that continuant is [5]:

$$\begin{vmatrix} S_{2l+1} & T_{2l+1} & & & \\ R_{2l+2} & S_{2l+2} & T_{2l+2} & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & R_{n+2l} & S_{n+2l} & T_{n+2l} \\ & & & R_{n+2l+1} & S_{n+2l+1} \end{vmatrix} = 0\,;$$

$$R_k = -2(2mA/\hbar^2)^{1/2}(k - n - 2l - 2)$$

$$S_k = l\left(\frac{2mB^2}{A\hbar^2}\right)^{1/2} + \frac{2mD}{\hbar^2} - \left(\frac{2mB^2}{A\hbar^2}\right)^{1/2} k$$

$$T_k = (k+1)(k-2l)\,.$$

The first condition gives the energy levels which are found to be explicitly independent of D and the second condition gives the allowed D values for each value of l and n:

$$E = \sqrt{(2A\hbar^2/m)}\,(n + l + \tfrac{3}{2}) - B^2/4A\,.$$

2) *Cylindrical coordinates*: $r^2 = x^2 + y^2$. The

same theory holds $l(l+1)$ being simply replaced by μ^2 in eq. (1). It must be pointed out that eq. (3) is also true in this case provided one considers odd negative values of $c = (-2\mu + 1)$ in place of even values as in case 1.

Conclusion. Schrödinger's equation with potential $Ar^2 + Br - D/r$ is exactly soluble only for special values of D in both spherical and cylindrical coordinates. Classically the same problem leads to elliptic integrals. Theories encountered in the general case may degenerate into elementary integrations when D is well fitted leading to a stable periodic motion so that both classical and quantum descriptions are quite analogous.

References

[1] P. M. Morse and H. Feshbach, Methods of theoretical physics (McGraw Hill, 1953).

[2] M. S. Plesset, Phys. Rev. 41 (1932) 278.

[3] A. P. Hautot, J. Math. Phys., March issue 1972.

[4] T. Muir, A treatise on the theory of determinants (Dover S670) (1960).

[5] A. P. Hautot, Bull. Soc. Roy. Sci. Liège 38 (1969) 654.

* * * * *